Lagrange Multipliers

A method for solving optimization problems subject to a constraint, i.e., finding points \((x_0, y_0, z_0)\) at which \(g(x_0, y_0, z_0) = k\) and \(f(x_0, y_0, z_0)\) is greatest/least among these.

\(z = f(x, y) = 2x^2 + 4y^2\)

Constrained minimum

Constrained maximum

\((0, 1, 4)\)

\((1, 0, 2)\)

\(x^2 + y^2 = 1\)
Illustration of the idea in 2 variables

Find min/max of \( f(x,y) = xy \) on the ellipse \( x^2 + 4y^2 = 1 \).

- at extrema (dark dots) ellipse & level curves are tangent, so have same normal direction, i.e. \( \nabla g \parallel \nabla f \)

\[ \nabla f = \lambda \nabla g \]

\( \lambda \) is the "Lagrange multiplier"
So, we must solve a system of 3 equations:

1) \( f_x(x,y) = \lambda g_x(x,y) \)
2) \( f_y(x,y) = \lambda g_y(x,y) \)
3) \( g(x,y) = c \)  

\( (c=1 \text{ in example above}) \)

There are 3 unknowns: \( x \), \( y \), and \( \lambda \).

In our example,

1) \( y = \lambda (2x) \)
2) \( x = \lambda (8y) \)
3) \( x^2 + 4y^2 = 1 \)
Combine 1) with 2)

Check possible solutions against 3)

We obtain 4 solutions:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>( f(x, y) = xy )</th>
<th>classification</th>
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Note: If the question asked for min/max of \( f(x,y) = xy \) on the disk \( x^2 + 4y^2 \leq 1 \), then we would

1) Find critical points of \( f(x,y) \) and see if any lie inside \( x^2 + 4y^2 \leq 1 \).

2) Use the method of Lagrange Multipliers to find the possible max/min points on the boundary.

3) Compare values of \( f(x,y) \) at the points found in 1) & 2).

Note: There is no general method for solving the equations obtained by the Lagrange Multipliers method. It is important to do examples to get good at it.