MATHEMATICS 152, FALL 2008
THE MATHEMATICS OF SYMMETRY
Tips on Proofs and Presentations

Last revised: August 11, 2008

In this class you will be asked to present definition, theorems, proofs and examples at the blackboard: the benefits of learning to do this well go beyond the fame and adulation of your classmates.

1 The cycle of mathematics: play and distill.

There is a contrast between the formal presentation of mathematics and its practice. Mathematicians (including yourself) interact with mathematics through working out examples, formulating conjectures (guesses) and verifying calculations, sometimes even running experiments on a computer, or making a paper model. It is only when we have sufficiently muddied our hands in this 'playing' phase that we can step back and collect our thoughts. A mathematical text is like a finished skyscraper, or a completed symphony played at the correct tempo. But mathematics is not written in this complete form – like an other endeavour, learning and *doing* mathematics, involves, above all else, experimentation and play.

However, mathematics is written in quite a different, spare and linear fashion: Definition, Theorem, Proof, Definition, Theorem, Proof... When a student reads a mathematical text, he, she or it is set the task of discerning, between the beautiful polished logical arguments, their source and the route one follows to get to them. Your textbook for this course (and all good texts) try to aid the difficulty with interspersed exercises and philosophical musings, but the work is always, in the end, the reader's. Then why do we present mathematics in this polished definition-theorem-proof progression at all? Because it serves to *verify* and to *distill*. Only mathematics presented in a careful logical progression can be verified by others to be logically true. And the process of organising mathematics for a direct and logical presentation has the effect of clarifying which of the tools and pieces you have tossed about in the mud are actually essential. The clarity of perspective that results also allows us to know which questions to ask next.

And so it is a cycle: *play* (experiment, guess, try, calculate,. . .) followed by *distill* (verify, write, polish, . . .) followed again by *play* (experiment, guess, . . .) and so on and so on.[1]

For example, this course concerns itself, among other things, with *groups*, *permutations*, *symmetries*, and *geometries*. Each of these things has a mathematical

[1] This is really no different than any other creative endeavour, including drawing, composing, making skyscrapers, figuring out how to get what you want out of your mother, and, yes, laboratory science.

definition, which is very important to know. But we will largely understand these definitions through examples. Our intuition comes from playing with examples. The course itself, and mathematics in general, has *play-distill* macro- and micro-cycles. To play with a definition, for example, you will create examples which do or do not satisfy the definition. But each example on the crest of the play-wave of this macrocycle is itself a microcycle: to create an example you make some guesses and do some calculations and experiments (play) before writing down a short proof (distill) that it does or does not satisfy the definition. The very smallest of these micro-cycles are a natural process of the human brain and are barely discernible.[2]

When you are giving a presentation, you will be presenting a finished product: a definition or a theorem, or an example or calculation (you will hide all your mistakes and wrong turns). You will be 'hiding the scaffolding' when you do this, and the job of both students and teachers is to learn to climb that scaffolding ourselves. So you will have two seemingly contradictory goals: first, to present mathematics in a formally correct, polished and clear manner; and second, to share with the other students the underlying process of creation. This is your challenge.

2 General strategies for presentations.

When you have been assigned a topic, your first task is the 'play' phase. Go home, try out some calculations, attempt to work out some examples, and put all your ideas on scratch paper. When you think you have a solution, get a fresh sheet of paper and try to organise the thoughts into an order, using only those pieces of your experiments that seem essential. At the bottom, as a separate note, jot down some ideas about *how* you decided what to play with and how you discovered the key ideas. This can be very difficult: often we don't know how we have the insights we have. Even if you are not able to give a good explanation, the reflection is worth the effort: you will learn patterns of your own thinking.

Then, get another fresh sheet of paper. Now it is time to try your hand at exposition. Using your outline, try to write out everything as you would expect to see it in a textbook or on the blackboard. Then practice explaining what you have written out loud. Make sure that at this stage, you take the time to explain how you had the ideas you used, if you can.

Sometimes the task for a presentation will be to look up some definitions from the book and explain them to your classmates. It may seem that you can shortcut the process above by just copying the definitions out of the text onto the board. Not so! To understand definitions (or any logical argument in a book), you

[2] Play is the strategy human brains use naturally, but I'm often surprised to see that students suspend the play instinct when they meet mathematics. Therefore I will carry on about it for a while to convince you to turn it back on.

must play with them. Therefore, the 'play' phase in this case involves creating examples that do satisfy the definition, and examples that don't. Try to discern the purpose of the definition, and what may have inspired its author to create it. Check whether things you have encountered in the course before satisfy the definition or not. Think of questions the definition may cause you to ask. If the definition is similar to another definition you have seen, what is the difference? Are there questions from that context that could be applied here? If any of this 'play' results in interesting ideas, you can include them in your presentation.

These strategies should be applied to *all* your mathematical interactions in the course, not just preparing for presentations. When you read a proof in the text book, *play*. Figure out what happens if you apply the method of proof to something that doesn't satisfy the hypotheses of the theorem – where does it break down? Try to test the theorem on some examples. Try to see if you can shorten the proof, or make it better. See if you can discern the overall structure of the proof (give an outline). Experiment with applying the theorem to different examples, and see if it gives interesting consequences.

Mathematics is really just a playground for adults.

3 Specific strategies for presentations.

3.1 Boardwork and voice

- Face the class while speaking.
- Write clearly (writing slowly and carefully helps).
- Speaking and writing simultaneously is very difficult; you can pause to write, then turn to speak.
- Speak loudly and confidently, which keeps everyone awake and interested.
- If you are nervous, pause and take a deep breath.
- It is your choice whether to be formal, conversational, or casual in tone.

3.2 Exposition

- Consider pausing to ask for questions.
- Consider incorporating guesses or contributions from the class.
- Think of the moments where you were confused in solving the problem; warn the class away from possible misconceptions.

- Isolate the key points ahead of time and be sure to emphasise them during the presentation.
- Tell the class where you got your ideas (if you can tell).
- Consider telling the class what you tried that didn't work (briefly of course!).
- Do not be afraid to pause to collect your thoughts at any time. (Pauses seem much longer to you than to the audience.)
- Do not be afraid to tell me to stop interrupting you and give you a moment to collect your thoughts.
- Do not be afraid to argue with me. Argument is very useful. And I'm wrong plenty of times.
- Do not be afraid to argue with other students. They're also wrong on occasion.
- Any argument should be friendly.

3.3 Tips for the audience.

- Ask questions.
- Make comments you think may clarify or extend the topic.
- No tomatoes or particularly juicy projectiles, please.
- If you have a comment on presentation style, save it for the end, and be polite.
- Comments on presentation style can also be made anonymously and privately through emailing me.

4 Specific strategies for play.

4.1 Definition

- Find an example which does not satisfy the definition. What part breaks? Can you find an example that satisfies that part but breaks another part?
- Find an example which does satisfy the definition. What can you change about it and have it still satisfy the definition? What can you not change about it?

- Try some small variations on types of objects that the definition may or may not apply to. Can you come up with rules of the form: 'all objects of a specific type which satisfy the definition must have property X'? This is a conjecture. If you can prove it, it's a theorem.

- Compare it to definitions you've seen before. Are there objects which satisfy one but not the other? How about vice versa? Does one definition imply the other?

4.2 Theorem

A theorem is a true statement of the form *if A then B*. A is the *hypothesis* and B is the *conclusion*.

- Find something which satisfies the hypothesis, and check that it satisfies the conclusion. If it doesn't, then either you made a mistake (find it) or the theorem is wrong (not impossible).

- Find something which does not satisfy the hypothesis and check whether it satisfies the conclusion. If it does, could the theorem be extended? Can you use the theorem's proof to prove this object satisfies the conclusion or does it need a new proof?

- Figure out where in the proof each hypothesis is used. See if a weaker hypothesis could be used. Work through the proof with an example which fails the hypothesis and see how it falls apart at that point.

- Does the theorem remind you of any other theorem in another branch of mathematics? Much new mathematics is done by analogy in exactly this way.

4.3 Example

- Apply all the definitions you have to the example. Which does it satisfy?

- See if your example satisfies the hypotheses of any theorem, and check the conclusion.

- Work out explicitly the structure of the example, and make a table, graph or other presentation of it. Maybe a paper model.

- Look for patterns in the data you collect and make conjectures (guesses) about possible theorems.

- See if the patterns depend on certain properties of your example and try to construct examples with similar properties.

- See if you can make a definition that includes your example and similar examples and such that anything satisfying the definition has similar properties to your example.

4.4 Problem and proof

This is the most difficult to give specific strategies for. An excellent book is *Solve It* by George Polya.

- Look up all the relevant concepts for your problem and remind yourself of the fun you had playing with them (what did you do?).
- Play with these concepts together. For example, if your problem is to verify that an example satisfies a definition, see if you have a useful theorem like 'things with property A satisfy the definition.' and try to verify property A instead.
- Just putting all the pieces together in your head will allow them to interact. Relax and have fun. Allow yourself to be distracted by related problems and questions (don't do homework at the last minute). Ideas cavort and carry-on without your knowledge while you are asleep or doing other things. Starting your homework early gives plenty of time for this unconscious processing.

5 Some final words.

Make this course interactive. Work (play) with your classmates in section, work (play) with your CA and come to office hours. Make study groups with others in the course, discuss the course outside class hours and use the class discussion forums. Post reflections and ideas on the course blog (what did you play with that was fun or interesting?). Enjoy your chance to present, don't be nervous, and encourage everyone else. And above all, *have fun*.