MATHEMATICS 3110, SPRING 2019
SOME MOTIVATIONAL PROBLEMS IN NUMBER THEORY

KATHERINE E. STANGE

Number theory may be loosely defined as the study of the integers: in particular, the interaction between their additive and multiplicative structures. However, modern number theory is often described as the study of such objects as algebraic number fields and elliptic curves, which we have invented in order to answer elementary questions about the integers. Therefore, an argument can be made that the best way to define number theory is to exhibit some of these motivational problems.

0.1. **Are there infinitely many primes?** Yes, and you are invited to invent your own proofs of this fact (there are many). Here is one:

Due to Euler. By manipulation of geometric series,

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^{-1} = \prod_{p \text{ prime}} \left(\sum_{k=1}^{\infty} \frac{1}{p^k}\right) = \sum_{n=1}^{\infty} \frac{1}{n}.$$

I will explain the second equals sign in a moment. But if you believe the above, then here's the proof. The sum on the right diverges; this is impossible if the product on the left is finite.

I promised I would explain the second equals sign. This actually depends on the so-called *fundamental theorem of arithmetic*, which states that every integer has a unique factorization into prime numbers (up to sign). The terms on the right can be constructed by choosing terms from the factors on the left. For example, since $12 = 2^2 \cdot 3$, the term $1/12$ is constructed by choosing $1/4$ (that is, the $k = 2$ term) from the factor $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1 + 1/2 + 1/4 + 1/8 + \cdots$ corresponding to $p = 2$, then $1/3$ from the factor corresponding to $p = 3$, and then $1/1$ from the factor corresponding to $p = 5$ etc. The term $1/12$ will appear exactly once in the gigantic FOIL you perform to multiply out the product on the left. □

The nice thing about this proof is that it introduces the function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

at least for $s = 1$. This function is related to the primes because it is a product over primes:

$$\prod_{p \text{ prime}} \left(\sum_{k=1}^{\infty} \frac{1}{p^{sk}}\right) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

By introducing the variable s, we have built a sort of *generating function* for the prime numbers, called the *Riemann zeta function*. In other words, a function whose algebraic structure encodes information about what we want to study (the primes). If we let s be a *complex* variable, then we can consider this a complex function and we can ask all sorts of

Date: Last revised: January 15, 2019.

interesting questions about it. Among these is the question of *where are its zeroes?*[1]. It turns out to have some so-called 'trivial zeroes' on the real line, but it has others. The famous unsolved problem called the *Riemann Hypothesis* (one of the seven *Millenium Problems*) states that the non-trivial zeroes all have real part 1/2. Amazingly, the position of the zeros of the Riemann zeta function mirrors the positions of the prime numbers amongst the integers. I mean this very literally: there's a formula for the positions of the primes in terms of the positions of the zeroes (Riemann's Explicit Formula)!

The Riemann Hypothesis is considered one of the premier unsolved problems in modern mathematics, and most mathematicians both firmly believe the hypothesis and yet don't believe it will be proven in our lifetimes. It has so many powerful consequences in number theory, that the result must lie very deep. There are a great many research papers which prove results conditional on various forms of the Riemann Hypothesis; hundreds of results will suddenly be unconditionally true when a proof is eventually found.

The paradigm is that information about the zeta function translates to information about the primes:

1. Euler's proof says that since the zeta function diverges at $s = 1$, there are infinitely many primes.
2. If we can restrict the possible locations of zeroes of zeta enough, then we obtain a growth rate on the primes (the Prime Number Theorem, below).
3. If we know the Riemann Hypothesis, that the zeroes lie on the *critical line* of real part 1/2, then we know the growth rate of the primes to fantastic accuracy.

Use the notation $\pi(x)$ for the number of primes up to x. The Prime Number Theorem (Hadamard and De La Vallée Poussin, 1896) famously states that $\pi(x) \sim x/\log x$, or actually the slightly better approximation $\pi(x) \sim \operatorname{Li}(x) = \int_2^x \frac{dt}{\log t}$. The $\sim$ notation indicates that the ratio of the two functions tends to 1 in the limit. This growth rate, as a conjecture, goes back to Dirichlet and Gauss around 1800. Proofs of the prime number theorem all depended on complex analysis until a proof of Selberg and Erdös in 1949. By refining this theorem, we know that for some constant $c > 0$,

$$\pi(x) = \operatorname{Li}(x) + O(xe^{-c\sqrt{\log x}}).$$

The Riemann Hypothesis, in one form, gives a stronger error bound:

$$\pi(x) = \operatorname{Li}(x) + O(x^{1/2}\log x).$$

In both of these cases, we are using "big O" notation, and it means, for example in the latter case, that $|\pi(x) - \operatorname{Li}(x)|$ is eventually bounded above by a constant multiple of $x^{1/2}\log x$.

0.2. **Is there a closed formula for the n-th prime?** Believe it or not, there are some contenders, but they are not simple. Willans gives the following formula for the n-th prime p_n:

$$p_n = 1 + \sum_{m=1}^{2^n} \left\lfloor \sqrt[n]{n} \left(\sum_{x=1}^{m} \left\lfloor \cos^2 \pi \frac{(x-1)!+1}{x} \right\rfloor \right)^{-1/n} \right\rfloor,$$

which is certainly a closed formula in some sense. In fact, it is just a sort of obfuscation of the relationship between p_n and $\pi(x)$, using Wilson's theorem as a primality test:

[1] Strictly speaking, we need to use analytic continuation first, a method of complex analysis, to define it for the variable s over the entire plane of complex numbers.

Theorem 0.1 (Wilson's Theorem). *p is prime or 1 if and only if $(p-1)! \equiv -1 \pmod{p}$.*

Neither is it particularly useful for computation, so I would say it is not a very satisfactory answer.

0.3. **Is there a (possibly multivariate) polynomial that gives exactly all the primes when evaluated on all integer inputs?** No. However, there are multivariate polynomials whose *positive* values are exactly all the primes, as the variables range over *natural* numbers. Such a polynomial in 26 variables, due to Jones, Sato, Wada and Wiens, is

$$\begin{aligned}(k+2)(1 &- [wz+h+j-q]^2 - [(gk+2g+k+1)(h+j)+h-z]^2 - \\ &[16(k+1)^3(k+2)(n+1)^2+1-f^2]^2 - [2n+p+q+z-e]^2 - \\ &[e^3(e+2)(a+1)^2+1-o^2]^2 - [(a^2-1)y^2+1-x^2]^2 - \\ &[16r^2y^4(a^2-1)+1-u^2]^2 - [n+l+v-y]^2 - \\ &[(a^2-1)l^2+1-m^2]^2 - [ai+k+1-l-i]^2 - \\ &[((a+u^2(u^2-a))^2-1)(n+4dy)^2+1-(x+cu)^2]^2 - \\ &[p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m]^2 - \\ &[q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x]^2 - \\ &[z+pl(a-p)+t(2ap-p^2-1)-pm]^2)\end{aligned}$$

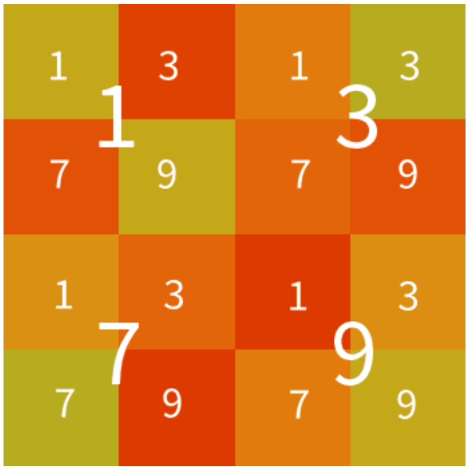
If you prefer fewer variables, you can get down to 10 variables if you let the degree go up to 15905. The proof is based on the logical notion of a Diophantine set.

0.4. **Are there infinitely many primes of the form $4n+1$?** Yes. More generally, there are infinitely many primes of the form $an+b$ for any coprime a and b. This is Dirichlet's celebrated theorem on arithmetic progressions (1837). We won't cover the proof (look to graduate school for that). In general, we expect about half of all primes to be congruent to 1 modulo 4 and the other half to be congruent to 3 mod 4. However, there are more in the former category, in the sense that, counting up to N, the former category is usually larger. This is called Chebyshev's Bias. See 'Prime Number Races' by Granville and Martin (American Mathematical Monthly).

0.5. **If you know the nth prime ends in a particular digit, is the final digit for the $(n+1)$-st prime equally likely to be any of the four possibilities $1,3,7,9$?** Something we only noticed this millenium: conjecturally/datawise NO. Primes don't seem to like to repeat final digits, so if p_n ends in 1, then p_{n+1} is less likely to end in 1. What? See *Unexpected Biases in the Distribution of Consecutive Primes* by Oliver and Soundararajan.

Some pictures illustrate the data, and in fact, the behaviour was noticed by drawing these pictures, adapting a visual analysis method used for DNA. DNA is a sequence with four possible digits, A,G,C, and T. In our case we use the sequence $p_n \pmod{10}$ (in other words, the remainder upon division by ten) which has four possible values, 1, 3, 5 and 7.

To draw the picture, break the square into four quadrants, labelled with your four digits. Then break each of these quadrants up into four sub-quadrants, labelled with the four digits. For example, in the leftmost picture, the upper-left box (of 16 such boxes) corresponds to the sequence "1, 1", indicating it is the 1-quadrant in the first subdivision, and then the 1-quadrant in the next subdivision. Then, colour that box according to the *frequency* of the corresponding two-digit sequence, as a subsequence of the sequence of primes modulo 10 up to some large number. A greenish colour means "1, 1" doesn't occur as often as the boxes which are red. For example, "9, 1" occurs much more often, as does "7, 9". For example, the boxes in the upper left demonstrate that a prime ending in 1 is less likely to be followed by a 1 than a 3. The picture at right is the same, but with longer sequences, meaning more subdivisions.



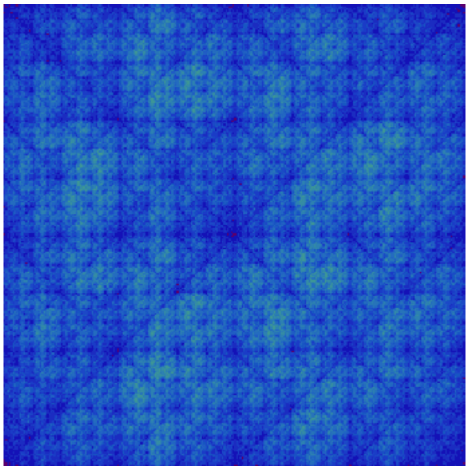
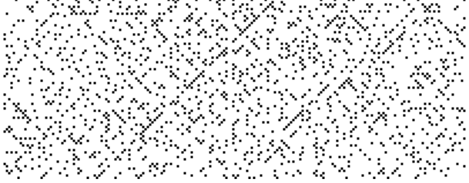
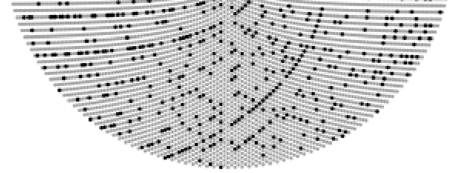


Image due to matthen.com.

0.6. **Are there infinitely many primes of the form $n^2 + 1$?** The more general question can be posed for any polynomial. It is unknown for any quadratic polynomial. Iwaniec has shown that there are infinitely many n for which $n^2 + 1$ is the product of at most two primes.

Ulam noticed that if you draw the primes in a spiral around the origin on a square grid, it looks far from random. To be a bit more precise, number the grid positions of the plane consecutively in a square spiral out from the origin. Then shade in the ones that are prime. The integers which are values of certain quadratic polynomials eventually head out along diagonal lines. The most visible diagonal on his spiral is $n^2 + n + 41$, which is prime for $0 \leq n < 40$ (but not for $n = 40$). There's a conjecture of Hardy and Littlewood about the density along these diagonals as it depends on the coefficients of the quadratic. Below is the Ulam spiral at left and the Sacks, a variation in which the diagonals become curved lines, at right.





0.7. **Are there infinitely many primes p for which $p+2$ is prime?** This is the famous twin primes conjecture (in the affirmative). It is still unsolved. More generally, one can ask how often $f_1(x)$ and $f_2(x)$ are simultaneously prime for some polynomials f_1 and f_2. Of course, there are infinitely many pairs of integers a, d such that a and $a+d$ are prime; we just don't know if we can take $d = 2$ infinitely often. In fact, van der Corput showed there are infinitely many 3-term arithmetic progressions in 1929. In 2004, Green and Tao showed there are infinitely many length k arithmetic progressions for all k.

Why do we think the answer is yes? This is an example of a pervasive heuristic argument in number theory. Using the Prime Number Theorem, we can guess that the 'probability' of a number x between 1 and N being prime is about $1/\log N$. Therefore, we expect the chance that both x and $x+2$ are prime is about $1/\log^2 N$: there will be about $N/\log^2 N$ twin prime pairs below N.

But wait, this also predicts that there are infinitely many primes p such that $p+1$ is prime! Refine the model: odd numbers between 1 and N have a $2/\log N$ chance of being prime, while even ones have a 0 chance. Refine it for multiples of 3, of 5, etc. and eventually we obtain a count of twin primes that is

$$2 \prod_{p \text{ odd prime}} \left(1 - \frac{1}{(p-1)^2}\right) \frac{N}{\log^2 N}.$$

(This is the *Hardy Littlewood* conjecture.) Not seeing any obvious reasons this is wrong, we conjecture this as the growth rate of twin primes. This relies on the oft-used heuristic that, having identified the 'obvious' ways in which primes are not random (congruence conditions, like most even numbers are not prime), they are otherwise *entirely random*! This is, of course, absurd.

Chen has shown in the 70's that there are infinitely many primes p such that $p+2$ is a product of at most two primes. This uses 'sieve methods'. In 2013, Yitang Zhang proved that there is some integer $N < 7 \times 10^7$ such that there are infinitely many prime pairs $(p, p+N)$. The bound on N was reduced to 246 in 2014 by a Polymath Project led by Tao.

0.8. **Up to N, are there always more natural numbers with an odd number of prime factors than with an even number of prime factors?** This is known as the Pólya Conjecture, and it seems heuristically reasonable that 'most' integers have an odd number of prime factors. It has important consequences in number theory and was widely believed between 1919 (when the conjecture was made) and 1958, when Haselgrove showed that it is false for infinitely many N. It is true until $N = 906,150,257$, when it fails. Never trust numerical evidence.

0.9. **Does $x^2 - 1141y^2 = 1$ have any integer solutions? (Note: if you ask the computer to check up to 25 digits, it will find none.)** Another case of misleading numerical evidence. The first solution to $x^2 - 1141y^2 = 1$ has y of 26 digits; there are infinitely many solutions. This is an example of a Pell equation. For more examples of 'The Strong Law of Small Numbers' (don't trust them), see Richard Guy's article by the same name.

The Pell equation is an example of a *Diophantine equation*. A Diophantine equation is just a polynomial equation for which we are interested in integer solutions. For example, $x^2 + y^2 = z^2$, in calculus, describes a *cone*. As a Diophantine equation, we ask about its integer solutions, meaning solutions where x, y and z are integers. These are called

Pythagorean triples (the most famous being 3, 4, 5), and there are infinitely many of them. In fact, they have a parametrization, meaning they are exactly the triples of the form

$$(p^2 - q^2, 2pq, p^2 + q^2)$$

(together with their reorderings), as p and q vary through all *relatively prime* pairs of integers. That is, p and q can be chosen to be any integers which don't share a common prime factor. To get 3, 4, 5, for example, choose $p = 2$, $q = 1$.

0.10. **Does $x^3 - y^2 = 1$ have any integer solutions besides $(1, 0)$? (Note: if you ask the computer to check up to 25 digits, it will find none.)** No. This is an example of an elliptic curve, and all elliptic curves have finitely many integral solutions. More generally, equations in two variables are called *curves* (after their appearance if we graph them). They fall into different classes depending on their shapes when graphed as complex surfaces. Complex surfaces are classified by genus, which is the number of holes. Genus 0 means no holes: a ball. Genus 1 is an elliptic curve, which is like a donut, or bagel (one hole). Genus 2 is like a donut with two holes, etc. The behaviour of their integer or rational points depends on this classification via their complex shape: this is a surprise, since we are only asking about their integer solutions! Genus 1 curves (and higher genus) always have only finitely many integer solutions. But even knowing it is finite, it is hard to figure out how many solutions there are. It's a bit of work to prove that $x^3 - y^2 = 1$ has no other solutions. (Note that the Pell equation in the last question is genus 0 and has infinitely many integer points.)

0.11. **Does $x^n + y^n = z^n$ have any non-zero integer solutions for integers $n > 2$? (Note: if you ask the computer to check up to 25 digits, it will find none.)** No. We know it should be finite by the general theory (these are higher genus), but it is much harder to show there are no solutions at all. This is Fermat's famous Last Theorem (from 1637), for which Fermat famously claimed, in a note in a textbook, to have a proof "which this margin is too small to contain." Fermat's famous non-marginal solution is often presumed to have been an error. Fermat's Last Theorem has been one of the drivers behind the development of number theory, as well as the theory of elliptic curves and modular forms, which finally solved it. The objects of study in number theory are hidden deeply behind the simple problems which motivate the area.

0.12. **Is there an algorithm to determine if a given polynomial equation in any number of variables has an integer solution?** This is Hilbert's 10th Problem. Actually, he asked the audience to devise such a process, as it came as quite a surprise that the answer would be NO. This is a celebrated result of Davis, Matiyasevich, Putnam and Robinson. The existence of a polynomial whose positive values on natural numbers are all the primes is a corollary. The proof lies in the realm of logic, and uses facts about the Fibonacci numbers in an essential way. For a wonderful read, see the book "Hilbert's Tenth Problem," by Matiyasevich.

The same question for rationals, in place of integers, is an open problem.

0.13. **Are there any quadratic forms with integer coefficients which represent all positive integers? (A quadratic form is a degree two polynomial whose monomials are individually total degree two, for example $x^2 + 7y^2$.)** This is a bit of a trick question unless you include the stipulation that the forms be *positive definite*, i.e. do not take zero or negative values. Otherwise xy is such a form.

For positive definite forms, the answer is no, for *binary* and *ternary* forms (i.e. 2 and 3 variables). We will see a classification of which integers are the sum of two squares; this fundamental result goes back to Fermat in 1640, but a completely elementary proof is not very easy.

Lagrange showed in 1770 that every positive integer is the sum of four squares. For quaternary and higher, it has been proven by Bhargava and Hanke (the '290 theorem' in 2005), that to determine if a form is 'universal' in this manner, it suffices to determine if it represents $1, 2, \ldots, 290$. (This came after the '15 theorem' of Conway and Schneeberger which applies to so called 'matrix-integral' forms; i.e. forms whose non-diagonal coefficients are even.)

0.14. **Does there exist a polynomial-time algorithm to determine if n is prime? ('Polynomial time' means the time taken (or number of basic logical operations needed) grows polynomially in terms of the number of digits of n.)** A first method would be to check all divisors up to $\sqrt{n}$; this takes $O(\sqrt{n})$ time. At first glance, Fermat's Little Theorem seems a promising criterion.

Theorem 0.2 (Fermat's Little Theorem)**.** *For any prime p and integer a coprime to p,*

$$a^{p-1} \equiv 1 \pmod{p}.$$

However, many composite n also satisfy this equation for some a. If a composite n satisfies this equation for all coprime a, then it is called a *Carmichael number*, about which there are many questions (there are infinitely many such numbers).

In 1975, a deterministic polynomial time-algorithm was given by Miller, but this is only assuming the *Extended Riemann Hypothesis*[2]. Around the same time some randomized polynomial-time algorithms were discovered (meaning it can return NO when it should return YES, but with random probability $< 1/2$), and many more have appeared since. Finally, in 2002, Agrawal, Kayal and Saxena found the desired algorithm, running in $O(\log^{15/2} n)$ time. This is polynomial-time since the number of digits of n is $log(n)/log(2)$.

0.15. **Can we factor numbers in polynomial time?** A good reference on this extensive subject is the book "Prime Numbers: A Computational Perspective," by Crandall and Pomerance. The quick answer is that there are sub-exponential algorithms known since the 70's, but no polynomial time algorithms, even under various generalised Riemann Hypotheses. However, there does not seem to be any evidence indicating that it is not possible, besides the fact that we have tried and failed, especially since the 70's. However, there are a great many very interesting algorithms, some of which we may meet in this class. With current methods, we can factor integers up to about 232 decimal digits (it took two years / 2000 computing years in 2009).

What complexity class is it? $\mathcal{P}$ refers to problems for which there are deterministic polynomial time algorithms[3]. $\mathcal{NP}$ refers to problems for which a correct answer can be verified

[2] A note on the Extended Riemann Hypothesis. The terminology on the various extensions of the Riemann Hypothesis is confusing; see the book "The Riemann hypothesis: a resource for the afficionado and virtuoso alike," by Peter Borwein, Stephen Choi, Brendan Rooney and Andrea Weirathmueller. The version used here is the usual critical-strip statement, applied to some particular Dirichlet L-functions (but not all).

[3] $\mathcal{P}$ actually refers to decision problems, but you can accomplish factoring via a yes/no problem something like "does n have a divisor with k-th digit j?" etc.

in polynomial time. Because of the AKS primality testing algorithm of 2005 (see above), factoring is in $\mathcal{NP}$. Famously, we do not know if $\mathcal{P} = \mathcal{NP}$.

0.16. **For any irrational number α, are there infinitely many rational numbers p/q such that $|\alpha - p/q| < 1/q^2$?** True for all irrational α; this is an application of the pigeonhole principle due to Dirichlet. It is Fields Medal work[4] that for any algebraic α, there are only finitely many p/q such that $|\alpha - p/q| < 1/q^{2+\epsilon}$ (Roth's Theorem, 1955). This is the fundamental question of the area called *Diophantine approximation*.

[4] The Fields Medal is often explained as the 'Nobel Prize' of mathematics. There's no Nobel Prize in mathematics, for reasons which are unclear but often gossiped about.