Example

The example below illustrates the process for solving the Diophantine equations $12x + 22y = 4$ (in other words, finding all integer solutions).

1. Compute the $gcd(22, 12)$ using the Euclidean algorithm:

   
   \begin{align*}
   22 &= 1 \cdot 12 + 10 \\
   12 &= 1 \cdot 10 + 2 \\
   10 &= 5 \cdot 2 + 0
   \end{align*}

   You can skip this step if you are comfortable doing it in combination with the next step. But at this point we have learned that $gcd(22, 12) = 2$ (from looking at the second-to-last line).

2. The corresponding Extended Euclidean Algorithm:

   Notation: The card

   
   \begin{center}
   \begin{array}{c}
   (x, y) \\
   \text{gives} \\
   c
   \end{array}
   \end{center}

   tells you that the linear combination $12x + 22y = c$.

   In what follows, notice that I choose $(0, 1)$ or $(1, 0)$ on the first two cards depending on the position of $12$ and $22$ in the equation. Just make sure your starting cards are true facts!

   \begin{align*}
   (0, 1) \quad &\text{gives} \\
   (1, 0) \quad &\text{gives} \\
   (-1, 1) \quad &\text{gives}
   \\
   22 &\quad = 1 \cdot \\
   12 &\quad = 2 \cdot \\
   10 &\quad = 2 \cdot
   \end{align*}

   \begin{align*}
   (1, 0) \quad &\text{gives} \\
   (-1, 1) \quad &\text{gives} \\
   (2, -1) \quad &\text{gives}
   \\
   12 &\quad = 1 \cdot \\
   10 &\quad = 2 \cdot \\
   2 &\quad = 2 \cdot
   \end{align*}

   \begin{align*}
   (-1, 1) \quad &\text{gives} \\
   (1, 0) \quad &\text{gives} \\
   (-11, 6) \quad &\text{gives}
   \\
   10 &\quad = 2 \cdot \\
   10 &\quad = 2 \cdot \\
   0 &\quad = 0
   \end{align*}

   3. Check that I didn’t make arithmetic errors, by making sure the final card is correct:

   
   \begin{align*}
   12(-11) + 22(6) &= 0.
   \end{align*}

   4. Collect the fact from the second to last card on the right column:

   
   \begin{align*}
   12(2) + 22(-1) &= 2.
   \end{align*}
5. To obtain a solution to our original problem, we multiply by an appropriate multiple (in this case, 2):

\[ 12(4) + 22(-2) = 4. \]

So \(12x + 22y = 4\) has at least one solution, \((4, -2)\).

6. Find all solutions to the homogeneous linear Diophantine equations, i.e. the full set of solutions to \(12x + 22y = 0\) is (by the theorem)

\[
\left\{ \left( \frac{-22k}{\gcd(12, 22)}, \frac{12k}{\gcd(12, 22)} \right) : k \in \mathbb{Z} \right\} = \{(-11k, 6k) : k \in \mathbb{Z}\}.
\]

7. The full set of solutions to \(12x + 22y = 4\) is obtained by combining the homogeneous solutions with the one particular solution we found:

\[
= \{(4 - 11k, -2 + 6k) : k \in \mathbb{Z}\} = \{\ldots, (15, -8), (4, -2), (-7, 4), \ldots\}.
\]