The Phrasing of Definitions in Mathematics

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In mathematical writing, the phrasing of a definition is usually very formulaic. This is to prevent confusion.

The basic structure is very often the following.

Definition 1. *A ⟨ TYPE OF OBJECT ⟩ N is called ⟨* NAME *⟩ if ⟨ PROPERTY of A ⟩.*

The name is generally italicized or underlined (or, in the LaTeX definition environment, it is un-italicized), to indicate more clearly the term we are defining. For example,

Definition 2. *An integer N is called* even *if N is divisible by* 2*.*

It is also possible to invert this structure and provide the name later.

Definition 3. *Suppose N is an integer and that N is divisible by* 2*. Then we say that N is even.*

or

Definition 4. *Let N be an integer, and suppose N is divisible by* 2*. Then we say that N is even.*

or other variations on how to *let* or *suppose* the existence of A and the fact that it satisfies certain properties.

You may wish to add an addition sentence about notation (see the example below).

The bottom line

In all of these variations, the form is identical to a theorem, i.e. *If N is a certain kind of thing, then we call it a* name*.* A definition requires the same level of precision as a theorem statement.

The weird thing is that in reality, the theorem ought to be "if and only if". If an integer is divisible by two, we call it even. But also, if we call an integer even, it is divisible by two! (Otherwise we are lying, right?)

Strangely, it is only one direction of this implication that is written in mathematical definition-writing. This is weird, but it's the way it is.

Examples

Here are some examples.

Definition 5. *Let $a, b \in \mathbb{Z}$. A non-negative integer g is called the* greatest common divisor of a and b *if it satisfies the following properties:*

1. *$g \mid a$ and $g \mid b$*
2. *for any other integer d, if $d \mid a$ and $d \mid b$ then $d \mid g$.*

In this case, we write $g = \gcd(a, b)$.

The properties themselves are stated and written mathematically just as they would be in a theorem.