Axioms of the Integers

The integers (denoted $\mathbb{Z}$) are a set with the following properties:

1. (Operations) There are binary operations $+$ and $\cdot$, which takes pairs of elements of $\mathbb{Z}$ to elements of $\mathbb{Z}$.

2. (Commutativity) For all $a, b \in \mathbb{Z}$,
$$a + b = b + a, \quad a \cdot b = b \cdot a.$$

3. (Associativity) For all $a, b, c \in \mathbb{Z}$,
$$a + (b + c) = (a + b) + c, \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

4. (Identity) There are elements $0, 1 \in \mathbb{Z}$ so that
$$a + 0 = a, \quad a \cdot 1 = a.$$

5. (Additive inverses) For any $a \in \mathbb{Z}$, there exists $-a \in \mathbb{Z}$ so that
$$a + (-a) = 0.$$

6. (Positive Integers) There is a subset $\mathbb{P}$ of $\mathbb{Z}$ which we call the positive integers, and we write $a > b$ when $a - b \in \mathbb{P}$.

7. (Positive closure) For any $a, b \in \mathbb{P}$, $a + b, a \cdot b \in \mathbb{P}$.

8. (Trichotomy) For every $a \in \mathbb{Z}$, exactly one of the the following holds:
    - $a \in \mathbb{P}$
    - $a = 0$
    - $-a \in \mathbb{P}$

9. (Well-ordering) Every non-empty subset of $\mathbb{P}$ has a smallest element.

The Peano Axioms of the Non-negative Integers

The non-negative integers (denoted $\mathbb{N}$) are a set with the following properties:

1. There is a distinguished element 0.
2. There is a *successor function* $s : \mathbb{N} \to \mathbb{N}$.
3. The image of s does not contain 0, i.e. there is no element $n \in \mathbb{N}$ such that $s(n) = 0$.
4. The function s is injective, i.e. for all $x, y \in \mathbb{N}$, if $x \neq y$ then $s(x) \neq s(y)$.
5. (Induction.) Suppose $S \subseteq \mathbb{N}$. Suppose that $0 \in S$ and whenever $n \in S$, then $s(n) \in S$. Then $S = \mathbb{N}$.

The integers are then built out of the non-negative integers.