Math 3110: Quiz #2 – Solutions

February 24, 2019

Name:

Question 1

(16 minutes / 16 points) Short answers. Each question is worth 2 points.

1. Give the definition of a prime number.

   Solution. An integer $n > 1$ is *prime* if it has no divisors d such that $1 < d < n$.

   Note: many of you asserted the only divisors were 1 and n, but -1 and $-n$ are also divisors.

2. (True/False) The numbers $1/\sqrt{2}$ and $\sqrt{2}$ are commensurable.

   Solution. Their ratio is $\sqrt{2}/(1/\sqrt{2}) = \sqrt{2}^2 = 2$, which is rational. Therefore they are commensurable: *True*.

3. List two fractions, in lowest form, which are kissing $3/4$.

   Solution. a/b is kissing $3/4$ if $4a - 3b = \pm 1$. We can solve this by inspection: e.g. $a = 1$, $b = 1$. Another solution can be obtained by using the homogeneous solution, e.g. $a = 1 + 3 = 4$, $b = 1 + 4 = 5$. Therefore two kissing fractions are $1/1$ and $4/5$.

   The full list of fractions I saw while grading was $1/1$, $2/3$, $4/5$, $5/7$ and $7/9$, although there are infinitely many possibilities. Since it was requested in lowest form, $-1/-1$ doesn't qualify.

4. Compute $\sigma_0(81)$.

   Solution. This is the divisor-counting function. The factorization of 81 is $81 = 3 \cdot 3 \cdot 3 \cdot 3$, so its positive divisors are $1, 3, 9, 27, 81$. There are five of them, so $\sigma_0(81) = 5$.

5. Give the prime factorization of 225/7.

   Solution. The prime factorization is $225/7 = 3^2 \cdot 5^2 \cdot 7^{-1}$.

6. Circle the functions $f(x)$ below which satisfy $\lim_{x\to\infty} \pi(x)/f(x) = 1$. (Here, $\pi(x)$ is the prime counting function, the number of primes p in the range $1 < p < x$.)

$$\ln x \qquad \frac{x}{\ln x} \qquad \frac{\ln x}{x} \qquad \int_2^x \frac{dt}{t} \qquad \sin(x)$$

   Solution. The 2nd item should be circled. The first grows much too slowly; the third and fifth stay ≤ 1. The fourth is a version of the first (it should have a $\ln t$ in the denominator to become $li(x)$, which would be correct). The fact that the 2nd works is a fact we discussed (but did not prove) in class. The proof is challenging.

7. (True/False) There exist two irrational numbers x and y so that $x + y$ is rational, but $x - y$ is irrational.

   Solution. True. For example, $x = -y = \sqrt{2}$. Then $x + y = 0$ but $x - y = 2\sqrt{2}$.

8. Compute the mediant of 2/3 and 3/4.

   Solution. The mediant is given by $(2 + 3)/(3 + 4) = 5/7$.

Question 2

(10 minutes / 10 points)
Prove the following theorem.

Theorem 1. *There are infinitely many prime numbers.*

Solution Your text has a nice proof, as do your course notes. (Euclid's proof is the standard.) There are dozens of known proofs out there.

Question 3

(10 minutes / 10 points)

Prove the following theorem.

Hint: I **strongly** suggest using the prime factorization characterization of the gcd and lcm; this proof is much easier that way. If you've forgotten the characterization of the gcd, may I remind you with a hint: the power of a prime dividing $\gcd(a, b)$ is the *smaller* of its power in a and its power in b. You may use the prime factorization characterization of gcd and lcm without proof.

Theorem 2. *Let a, b be positive integers. Then* $\gcd(a, b) \operatorname{lcm}(a, b) = ab$.

Solution. Let a and b be positive integers. Then both have a prime factorization. We will write this as

$$a = \prod_p p^{e_p}, \quad b = \prod_p p^{f_p}.$$

Then, the gcd and lcm of a and b have prime factorizations given in terms of those written above, namely,

$$\gcd(a, b) = \prod_p p^{\min\{e_p, f_p\}}, \quad \operatorname{lcm}(a, b) = \prod_p p^{\max\{e_p, f_p\}}.$$

Then, we have

$$\begin{aligned} \gcd(a, b) \operatorname{lcm}(a, b) &= \prod_p p^{\min\{e_p, f_p\}} \cdot \prod_p p^{\max\{e_p, f_p\}} \\ &= \prod_p p^{\min\{e_p, f_p\} + \max\{e_p, f_p\}} \\ &= \prod_p p^{e_p + f_p} \\ &= \prod_p p^{e_p} \prod_p p^{f_p} \\ &= ab. \end{aligned}$$

Question 4

(10 minutes / 10 points)

Find four distinct Pythagorean triples (a, b, c) so that $0 < a < b < c$ and $\gcd(a, b) = 1$. You may use any method, but the method must be justified (i.e. not "guessing").

Solution. We have seen a bijection between rational slopes and Pythagorean triples. It involved the formula

$$m \to (m^2 - 1, 2m, m^2 + 1).$$

1. Letting $m = 2$, we obtain $3, 4, 5$. This qualifies.
2. Letting $m = 3$, we obtain $8, 6, 10$. This doesn't qualify; in fact it is a multiple/permutation of the first one.
3. Letting $m = 4$, we obtain $15, 8, 17$. Re-ordering, this is $8, 15, 17$ and it qualifies.
4. Letting $m = 5$, we obtain $24, 10, 26$. Dividing by 2 and reordering, this is $5, 12, 13$. This qualifies.
5. Letting $m = 6$, we obtain $35, 12, 37$. Re-ordering, this is $12, 35, 37$, and this qualifies.

We have now found four triples:

$$(3, 4, 5), \quad (8, 15, 17), \quad (5, 12, 13), \quad (12, 35, 37).$$

There are other possible solutions. It is possible, for example, to use rational m with denominator, but calculations by hand are harder, e.g. letting $m = 1/2$, we obtain $-3/4, 1, 5/4$, which, when scaled to become integers, is $(-3, 4, 5)$; changing sign on the first entry we obtain $(3, 4, 5)$, which we got from $m = 2$ with less computation.

Solution, second method.

For those who watched the video assigned on the website, some used that method, which is computationally slicker but involves talking about complex numbers. The idea is to pick a complex point $a + bi$, with $a, b \in \mathbb{Z}$, and square it. Then its distance from the origin becomes an integer c. Check out the video (under Lectures), for more explanation. For example, $(2 + 3i)^2 = -5 + 12i$ and this has length $\sqrt{5^2 + 12^2} = 13$. We obtain triple $(5, 12, 13)$.

The full list of valid triples I saw on the exams was

$$(3, 4, 5), (5, 12, 13), (8, 15, 17), (20, 21, 29), (7, 24, 25), (9, 40, 41), (16, 63, 65), (12, 35, 37), (11, 60, 61), (13, 84, 85).$$