Elliptic nets

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Elliptic divisibility sequences

Definition

$$W_{m+n}W_{m-n}W_1^2 = W_{m+1}W_{m-1}W_n^2 - W_{n+1}W_{n-1}W_m^2.$$

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A sequence *W* is an *elliptic divisibility sequence* if for all positive integers m > n,

$$W_{m+n}W_{m-n}W_1^2 = W_{m+1}W_{m-1}W_n^2 - W_{n+1}W_{n-1}W_m^2.$$

• Generated by W_1, \ldots, W_4 via the recurrence.

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- Example: 1,2,3,4,5,6,7,8,9,10,...

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- Example: 1, 3, 8, 21, 55, 144, 377, 987, 2584, 6765, ...

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- Generated by W_1, \ldots, W_4 via the recurrence.
- Example: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ...
- Example: 1, 3, 8, 21, 55, 144, 377, 987, 2584, 6765, ...
- Example: 1, 1, -3, 11, 38, 249, -2357, 8767, 496036, -3769372, -299154043, -12064147359, ...

Divisibility and Integrality

If W_1, \ldots, W_4 are integer with $W_1 = 1$, $W_2 W_3 \neq 0$, and $W_2 | W_4$, then the sequence ...



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; and

3. if $gcd(W_3, W_4) = 1$, it satisfies the Strong Divisibility Property

 $W_{gcd(m,n)} = gcd(W_m, W_n)$.

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Somos sequences

A Somos-k sequence is a sequence satisfying the recurrence

$$C_n C_{n+k} = \sum_{j=1}^{[k/2]} C_{n-j} C_{n-(k-j)}.$$

But in this talk, we will mean more generally with coefficients allowed, so

$$C_n C_{n+k} = \sum_{j=1}^{[k/2]} a_j C_{n-j} C_{n-(k-j)}.$$

 EDS are Somos-4, Somos-5, Somos-6, etc. (van der Poorten, Swart, 2004)

Elliptic nets

Connections

$$P = (0, 0)$$



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[2] $P = (3,5)$



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Example: $y^2 + y = x^3 + x^2 - 2x$, P = (0, 0)

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$$P = (0,0) \qquad 1$$

$$[2]P = (3,5) \qquad 1$$

$$[3]P = \left(-\frac{11}{3^2}, \frac{28}{3^3}\right) \qquad 3$$

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Example: $y^2 + y = x^3 + x^2 - 2x$, P = (0, 0)

$$\begin{split} P &= (0,0) &+ 1 \\ & [2]P &= (3,5) &+ 1 \\ & [3]P &= \left(-\frac{11}{3^2},\frac{28}{3^3}\right) &- 3 \\ & [4]P &= \left(\frac{114}{11^2},-\frac{267}{11^3}\right) &+ 11 \\ & [5]P &= \left(-\frac{2739}{38^2},-\frac{77033}{38^3}\right) &+ 38 \\ & [6]P &= \left(\frac{89566}{249^2},-\frac{31944320}{249^3}\right) &+ 249 \\ & [7]P &= \left(-\frac{2182983}{2357^2},-\frac{20464084173}{2357^3}\right) &- 2357 \end{split}$$

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$$P = (0,0) \qquad W_1 = +1$$

$$[2]P = (3,5) \qquad W_2 = +1$$

$$[3]P = \left(-\frac{11}{3^2}, \frac{28}{3^3}\right) \qquad W_3 = -3$$

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Sequences from division polynomials

Consider a point P = (x, y) and its multiples on an elliptic curve $E : y^2 = x^3 + Ax + B$:

 $P, [2]P, [3]P, [4]P, \ldots$

Sequences from division polynomials

Consider a point P = (x, y) and its multiples on an elliptic curve $E : y^2 = x^3 + Ax + B$. Then

$$[n]P = \left(\frac{\phi_n(P)}{\Psi_n(P)^2}, \frac{\omega_n(P)}{\Psi_n(P)^3}\right)$$

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where

$$\begin{split} \Psi_1 &= 1, \qquad \Psi_2 = 2y, \\ \Psi_3 &= 3x^4 + 6Ax^2 + 12Bx - A^2, \\ \Psi_4 &= 4y(x^6 + 5Ax^4 + 20Bx^3 - 5A^2x^2 - 4ABx - 8B^2 - A^3), \end{split}$$

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It gives an elliptic divisibility sequence of division polynomials.

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It gives an elliptic divisibility sequence of division polynomials. If we evaluate at P, we get the elliptic divisibility sequence associated to E and P.

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Division polynomials as complex elliptic functions

The *n*-th division polynomial has divisor

$$\sum_{P\in E[n]} (P) - n^2(\mathcal{O}).$$

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In complex case, fix a lattice $\Lambda \in \mathbb{C}$ corresponding to an elliptic curve *E*. For each $n \in \mathbb{Z}$, define a function Ω_n on \mathbb{C} in the variable *z*:

$$\Omega_n(z;\Lambda) = \frac{\sigma(nz;\Lambda)}{\sigma(z;\Lambda)^{n^2}}$$

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Division polynomials and sequences over finite fields

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Example

$$E: y^2 + y = x^3 + x^2 - 2x$$
 over \mathbb{F}_5 .
 $P = (0,0)$ has order 9.

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Example

 $E: y^2 + y = x^3 + x^2 - 2x \text{ over } \mathbb{F}_5.$ P = (0,0) has order 9.The associated sequence is $0, 1, 1, 2, 1, 3, 4, 3, 2, 0, 3, 2, 1, 2, 4, 3, 4, 4, 0, 1, 1, 2, 1, 3, 4, \dots$

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Example

 $E: y^2 + y = x^3 + x^2 - 2x \text{ over } \mathbb{F}_5.$ P = (0,0) has order 9.The associated sequence is $0, 1, 1, \frac{2}{2}, 1, 3, 4, 3, 2, 0, 3, 2, 1, 2, 4, 3, 4, 4, 0, 1, 1, 2, 1, 3, 4, \dots$

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The question - upping the dimension

The elliptic divisibility sequence is associated to the sequence of points [n]P on the curve.

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The question - upping the dimension

The elliptic divisibility sequence is associated to the sequence of points [n]P on the curve.

$$[n]P \leftrightarrow W_n$$

We might dream of ...

$$[n]P + [m]Q \leftrightarrow W_{n,m}$$

Or even ...

$$[n]P + [m]Q + [t]R \leftrightarrow W_{n,m,t}$$

etc.

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History of this question

- Robbins forum discussions of 'denominators' (c. 2001): Noam Elkies, Michael Somos, James Propp.
- Graham Everest, Peter Rogers, Thomas Ward, Nelson Stephens considered when these denominators may be prime (2002).

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Definition of an elliptic net

Definition (S)

Let *R* be an integral domain, and *A* a finite-rank free abelian group. An *elliptic net* is a map $W : A \rightarrow R$ such that the following recurrence holds for all *p*, *q*, *r*, *s* \in *A*.

$$egin{aligned} & W(p+q+s)W(p-q)W(r+s)W(r) \ & + W(q+r+s)W(q-r)W(p+s)W(p) \ & + W(r+p+s)W(r-p)W(q+s)W(q) = 0 \end{aligned}$$

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- Elliptic divisibility sequences are a special case (A = Z)
- In this talk, we will mostly discuss rank 2: $A = \mathbb{Z}^2$.

The octahedron recurrence

In the case of p = (l, m - 1, n), q = (1, 1, 0), r = (0, 1, 1), s = (0, -2, 0) for example, we obtain a recurrence of the form

$$aW(l+1, m, n)W(l-1, m, n) + bW(l, m+1, n)W(l, m-1, n) + cW(l, m, n+1)W(l, m, n-1) = 0$$

called the Octahedron Recurrence or Hirota Bilinear Equation. (Hirota (1981), subsequently many people, including David Speyer.)

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Laurentness

Theorem (S)

The terms of an elliptic net are generated by the recurrence relation from a finite set of initial terms. Furthermore, the terms are Laurent polynomials in a set of initial terms of size 4 for rank one, and size no larger than $3^n - 1$ for rank n > 1.

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The terms of an elliptic net are generated by the recurrence relation from a finite set of initial terms. Furthermore, the terms are Laurent polynomials in a set of initial terms of size 4 for rank one, and size no larger than $3^n - 1$ for rank n > 1.

Proof.

A lot of induction.

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Laurentness in rank one

In rank one, the terms are polynomials in initial conditions, with W(2) and W(1) possibly appearing to negative powers: If $a = W_1$, $b = W_2$, $c = W_3$, $d = W_4$, then

$$W_5 = rac{db^3 - ac^3}{a^3}, \qquad W_6 = rac{-a^4cd^2 - c^4b^2a + dcb^5}{ba^5},$$
 etc.

Laurentness in rank two

Theorem (S.)

Let $W : \mathbb{Z}^2 \to R$ be an elliptic net. All terms are polys with \mathbb{Z} -coeffs in variables

 $\frac{W(1,1),W(1,0),W(0,1),W(1,1)^{-1},W(1,0)^{-1},W(0,1)^{-1},}{W(2,1),W(1,2),W(2,0),W(0,2),}\\\frac{W(0,2)W(2,1)W(1,0)-W(0,1)W(2,0)W(1,2)}{W(0,1)^{3}W(2,1)-W(1,0)^{3}W(1,2)}.$

Integer terms

In particular, if

- W(1,0) = W(0,1) = W(1,1) = 1,
- the terms W(2,0), W(0,2), W(1,2), W(2,1) are integers and
- W(2,1) W(1,2) divides W(0,2)W(2,1) - W(2,0)W(1,2),

then all terms of the elliptic net are integers. e.g.

$$W(2,3) = W(0,2) \left(\frac{W(0,2)W(2,1) - W(2,0)W(1,2)}{W(2,1) - W(1,2)} \right) \\ - W(1,2)^2 W(2,1).$$

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Laurentness in higher rank

Theorem

$$S_n = \{\mathbf{v} \in \mathbb{Z}^n : \max_{i=1,\ldots,n} |v_i| = 1\},\$$

 $S'_n = S_n \cap \{ \mathbf{v} \in \mathbb{Z}^n : v_i = 0 \text{ for at least one } i \}.$

The terms of an elliptic net of rank n are Laurent polynomials in the following variables and coefficients:

1. *For n* = 3:

Variables: { $W(\mathbf{v}) : \mathbf{v} \in S_3$ }; *Coefficients:* \mathbb{Z}

2. *For* $n \ge 4$ *:*

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$$_{\circ}$$
 [3]*Q* $_{\circ}$ [1]*P* + [3]*Q* $_{\circ}$ [2]*P* + [3]*Q*

$$_{\circ}$$
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$$[1]Q$$
 $[1]P + [1]Q$ $[2]P + [1]Q$

。[1]*P*

 ∞ 0

$$[1]Q$$
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$$[1]Q \qquad [1]P + [1]Q \quad [2]P + [1]Q$$

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$$[1] \mathbf{P} \perp [1] \mathbf{O} \qquad [2] \mathbf{P} \perp [$$

$$[1]P + [1]Q$$
 $[2]P + [1]$

$$P + [1]Q \circ [2]P + [1]$$

。[2]*P*

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$$\begin{array}{c} \left(\frac{56}{25},\frac{371}{125}\right) \\ \circ\end{array} \quad \left(-\frac{95}{64},\frac{495}{512}\right) \\ \circ\end{array} \quad \left(\frac{328}{361},-\frac{2800}{6859}\right)$$

$$\circ \quad \left(\frac{6}{1}, -\frac{16}{1}\right) \qquad \circ \quad \left(\frac{1}{9}, -\frac{19}{27}\right) \qquad \circ \quad \left(\frac{39}{1}, \frac{246}{1}\right)$$

$$\circ \quad \left(\frac{1}{1}, \frac{0}{1}\right) \qquad \circ \quad \left(-\frac{2}{1}, -\frac{1}{1}\right) \qquad \circ \quad \left(\frac{5}{4}, -\frac{13}{8}\right)$$

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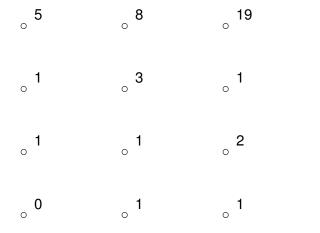
$$\ \ \, _{\circ} \ \ \, \left(\frac{56}{5^2},\frac{371}{5^3}\right) \quad \ \ \, _{\circ} \ \ \left(-\frac{95}{8^2},\frac{495}{8^3}\right) \quad \ \ \, _{\circ} \ \ \left(\frac{328}{19^2},-\frac{2800}{19^3}\right) \\$$

$$\circ \quad \left(\frac{6}{1^2}, -\frac{16}{1^3}\right) \quad \circ \quad \left(\frac{1}{3^2}, -\frac{19}{3^3}\right) \quad \circ \quad \left(\frac{39}{1^2}, \frac{246}{1^3}\right)$$

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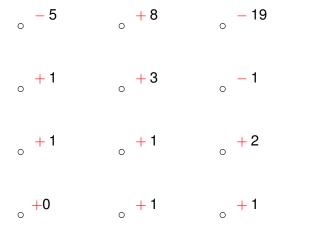
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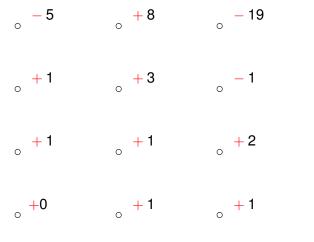
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An elliptic net!

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Curve + points give net

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Curve + points give net

Let *E* be an elliptic curve defined over a field *K*. For all $\mathbf{v} \in \mathbb{Z}^n$, we define rational functions $\Psi_{\mathbf{v}}$ on E^n which:

• form an elliptic net (in **v**)

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Curve + points give net

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Then for any fixed $\mathbf{P} \in E(K)^n$, the function $W : \mathbb{Z}^n \to K$ defined by

$$W(\mathbf{v}) = \Psi_{\mathbf{v}}(\mathbf{P})$$

is an elliptic net.

Connections

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Make functions

We know the divisor we want:

$$([v_1]P_1 + [v_2]P_2 = \mathcal{O}) - (v_1v_2)(P_1 + P_2 = \mathcal{O}) - (v_1^2 - v_1v_2)(P_1 = \mathcal{O}) - (v_2^2 - v_1v_2)(P_2 = \mathcal{O})$$

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As before, over complexes this allows us to define polynomials:

$$\Omega_{u,v}(z,w;\Lambda) = \frac{\sigma(uz + vw;\Lambda)}{\sigma(z;\Lambda)^{u^2 - uv}\sigma(z + w;\Lambda)^{uv}\sigma(w;\Lambda)^{v^2 - uv}}$$

Connections

Net polynomial examples

$$\Psi_{-1,1} = x_1 - x_2$$
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Net polynomial examples

$$\begin{split} \Psi_{-1,1} &= x_1 - x_2 \ , \\ \Psi_{2,1} &= 2x_1 + x_2 - \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 \ , \end{split}$$

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Theorem (S.)

There is a bijection of partially ordered sets:

 $\left\{ \begin{array}{l} \text{elliptic net} \\ W : \mathbb{Z}^n \to K \\ \text{modulo scale} \\ \text{equivalence} \end{array} \right\} \quad \leftrightarrow \quad \left\{ \begin{array}{l} \text{cubic Weierstrass curve C over K} \\ \text{together with m points in } C(K) \\ \text{modulo change of variables} \\ x' = x + r, y' = y + sx + t \end{array} \right\}$

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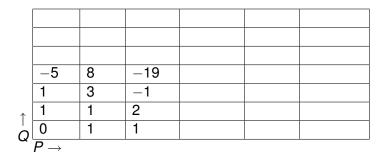
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- n = m and $W(\mathbf{v}) = \Psi_{\mathbf{v}}(P_1, \ldots, P_m, C)$
- explicit equations to go back and forth!
- singular cubics correspond to Lucas sequences or integers
- scale equivalence: $W \sim W' \iff W(\mathbf{v}) = f(\mathbf{v})W'(\mathbf{v})$ for $f: \mathbb{Z}^n \to K^*$ guadratic
- on left, remove nets with zeroes too close to the origin
- on right, remove cases with small torsion points or pairs which are equal or inverses

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| | 4335 | 5959 | 12016 | -55287 | 23921 | 1587077 |
|------------------|-----------------|------|-------|----------|-------|---------|
| | 94 | 479 | 919 | - 2591 | 13751 | 68428 |
| | - 31 | 53 | -33 | -350 | 493 | 6627 |
| | -5 | 8 | -19 | <u> </u> | - 151 | 989 |
| Ť | 1 | 3 | -1 | – 13 | -36 | 181 |
| | 1 | 1 | 2 | -5 | 7 | 89 |
| $\left \right $ | 0 | 1 | 1 | -3 | 11 | 38 |
| 9 | $P \rightarrow$ | | | | | |

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Fibonacci numbers

Consider the sequence of even-indexed Fibonacci numbers,

1, 3, 8, 21, 55, 144, 377, 987, 2584, 6765, 17711,...

satisfying W(n+2) = 3W(n+1) - W(n). Associated to: $y^2 + 3xy + 3y = x^3 + 2x^2 + x$, P = (0,0) (nodal).

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satisfying W(n+2) = 3W(n+1) - W(n). Associated to: $y^2 + 3xy + 3y = x^3 + 2x^2 + x$, P = (0,0) (nodal). Non-singular points of the curve isomorphic with $\overline{\mathbb{Q}}^*$:

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That is to say, C_{ns} is a twisted form of \mathbb{G}_m . The point P = (0,0) is associated to the unit $\left(\frac{3+\sqrt{5}}{3-\sqrt{5}}\right)$. Unit group in $\mathbb{Q}(\sqrt{5})$ is rank 1, so there's no interesting *rational* rank two Fibonacci numbers.

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Rank two Fibonacci numbers

Take another point $Q = (1, \sqrt{13} - 3)$ on this curve. The elliptic divisibility sequence associated to *C* and *Q* begins

1, $2\sqrt{13}$, 88, $576\sqrt{13}$, 97280, $2523136\sqrt{13}$, 1700790272, ...

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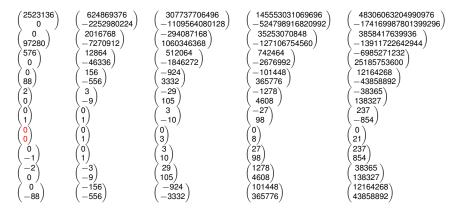
It's equivalent to the sequence sequence $A_n = \sqrt{2}^{n^2-1} W_{E,P}(n)$ beginning

1,
$$\frac{\sqrt{13}}{\sqrt{2}}$$
, $\frac{11}{2}$, $\frac{9\sqrt{13}}{2\sqrt{2}}$, $\frac{95}{4}$, $\frac{77\sqrt{13}}{4\sqrt{2}}$, $\frac{811}{8}$, $\frac{657\sqrt{13}}{8\sqrt{2}}$, $\frac{6919}{16}$, ...
satisfying $A_{n+2} = \left(\frac{\sqrt{13}}{\sqrt{2}}\right) A_{n+1} - A_n$.

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Rank two Fibonacci numbers

 $y^2 + 3xy + 3y = x^3 + 2x^2 + x, P = (0,0), Q = (1,\sqrt{13}-3)$



Where
$$\begin{pmatrix} a \\ b \end{pmatrix}$$
 means $a\sqrt{13} + b$.

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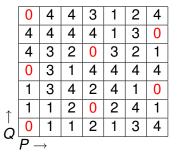
Example over \mathbb{F}_5

$$E: y^2 + y = x^3 + x^2 - 2x; P = (0,0), Q = (1,0)$$

| | 0 | 4 | 4 | 3 | 1 | 2 | 4 |
|-------------------|------------|---------------|---|---|---|---|---|
| | 4 | 4 | 4 | 4 | 1 | 3 | 0 |
| | 4 | 3 | 2 | 0 | 3 | 2 | 1 |
| | 0 | 3 | 1 | 4 | 4 | 4 | 4 |
| | 1 | 3 | 4 | 2 | 4 | 1 | 0 |
| ↑ | 1 | 1 | 2 | 0 | 2 | 4 | 1 |
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| G | <u>P</u> – | \rightarrow | | | | | |

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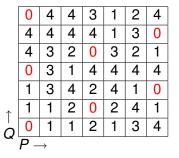
• The polynomial $\Psi_{\mathbf{v}}(\mathbf{P}) = 0$ if and only if $\mathbf{v} \cdot \mathbf{P} = 0$.

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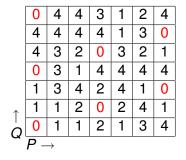


- The polynomial $\Psi_{\mathbf{v}}(\mathbf{P}) = 0$ if and only if $\mathbf{v} \cdot \mathbf{P} = 0$.
- These zeroes lie in a lattice: the *lattice of apparition* associated to prime (here, 5).

Connections

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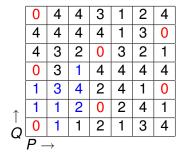
Periodicity property with respect to lattice of apparition



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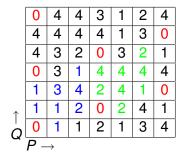
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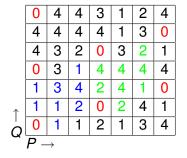
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Periodicity property with respect to lattice of apparition



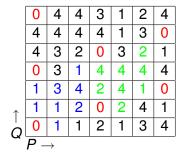
Periodicity property with respect to lattice of apparition



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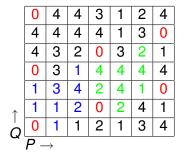
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• There are such translation properties.

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Translation properties

Let Γ be the lattice of apparition for an elliptic net W. Define $g: \Gamma \times \mathbb{Z}^n \to K^*$ by

$$g(\mathbf{r},\mathbf{m}) = rac{W(\mathbf{m}+\mathbf{r})}{W(\mathbf{m})}.$$

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Theorem (Ward n = 1; S., n > 1)

The function g is quadratic and affine linear in 2nd variable.

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Example

If n = 1, W(r) = 0, then

$$g(kr,m)=a^{mk}b^{k^2},$$

for all $k \in \mathbb{Z}$.

Elliptic curves and pairings For any divisor $(Q + S) - (S) \in Pic^{0}(E)$, we obtain an extension

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and $e_m : E[m] \times E[m] \rightarrow \mu_m$ defined by

$$e_m(P,Q) = \tau_m(P,Q)/\tau_m(Q,P)$$

is the Weil pairing (intersection pairing on homology of elliptic curve).

Elliptic nets and pairings

Theorem

Let Q_1, Q_2, Q_3 be points on an elliptic curve E and let W be any elliptic net associated to E and points $\mathbf{T} = (P_1, \ldots, P_n)$ such that we can find $\mathbf{q}_i \in \mathbb{Z}^n$ for which $\mathbf{q}_i \cdot \mathbf{T} = Q_i$ on the curve. The Tate-Lichtenbaum pairing of $Q_1 \in E[m]$ and $Q_2 \in E$ is given by

$$\tau_m(Q_1, Q_2) = \frac{W(m\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3)W(\mathbf{q}_3)}{W(m\mathbf{q}_1 + \mathbf{q}_3)W(\mathbf{q}_2 + \mathbf{q}_3)}$$

and the Weil pairing of $Q_1, Q_2 \in E[m]$ is given by

$$e_m(Q_1, Q_2) = \frac{W(m\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3)W(\mathbf{q}_1 + \mathbf{q}_3)W(m\mathbf{q}_2 + \mathbf{q}_3)}{W(m\mathbf{q}_1 + \mathbf{q}_3)W(\mathbf{q}_2 + \mathbf{q}_3)W(\mathbf{q}_1 + m\mathbf{q}_2 + \mathbf{q}_3)}.$$

(These formulæ are independent of q_3 and the choice of T.)

Periodicity and pairings

Reminder:

$$g(\mathbf{r},\mathbf{m})=rac{W(\mathbf{m}+\mathbf{r})}{W(\mathbf{m})}.$$



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Periodicity and pairings

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Combining our results, we have

$$au_m(P,Q) = rac{g(m\mathbf{p},\mathbf{q}+\mathbf{s})}{g(m\mathbf{p},\mathbf{s})},$$

and

$$e_m(P,Q) = rac{g(m\mathbf{p},\mathbf{q}+\mathbf{s})g(m\mathbf{q},\mathbf{s})}{g(m\mathbf{p},\mathbf{s})g(m\mathbf{q},\mathbf{p}+\mathbf{s})}.$$

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Primitive Divisors in Elliptic Divisibility Sequences

We may define a **Primitive Divisor** of a term W_n to be a prime p such that $p|W_n$ and $p \not| W_m$ for any 0 < m < n. We then have

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Theorem (Silverman's Elliptic Zsigmondy Theorem)

For every elliptic divisibility sequence there is a finite bound N such that for any n > N, W_n has a primitive divisor.

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There have since been many other results...

Elliptic nets

Connections

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Lattices of Apparition and Primitive Divisors

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Lattices of Apparition and Primitive Divisors

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Applications to Cryptography: Pairing computation

- Can calculate the terms of the sequence with a double-and-add algorithm.
- Thank you to Michael Scott, Augusto Jun Devigili and Ben Lynn for implementing the algorithm.

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| Algorithm: | Miller's | Elliptic Net |
|------------|-------------|--------------|
| type a | 19.8439 ms | 40.6252 ms |
| type f | 238.4378 ms | 239.5314 ms |

average time of a test suite of 100 randomly generated pairings in each of the two cases

Connections

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Applications to Cryptography: ECDLP

Problem

Let *E* be an elliptic curve over a finite field $K = \mathbb{F}_q$. Suppose one is given points $P, Q \in E(K)$ such that $Q \in \langle P \rangle$. Determine *k* such that Q = [k]P.

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Joint work with Kristin Lauter and performed at Microsoft Research.

EDS Discrete Log

Problem (Width s EDS Discrete Log)

Given an elliptic divisibility sequence W and terms W(k), $W(k+1), \ldots, W(k+s-1)$, determine k.

Elliptic nets

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Problem (EDS Association)

Determine $W_{E,P}(k)$ for the value of 0 < k < ord(P) such that Q = [k]P.

Problem (EDS Residue)

Determine the quadratic residuosity of $W_{E,P}(k)$ for the value of $0 < k < \operatorname{ord}(P)$ such that Q = [k]P.

Equivalence of problems

Theorem (S,L)

Let *E* be an elliptic curve over a finite field $K =_q$ of characteristic $\neq 2$. If any one of the following problems is solvable in probabilistic sub-exponential time, then all of them are:

- 1. ECDLP
- 2. EDS Association for non-perfectly periodic sequences
- 3. Width 3 EDS Discrete Log for perfectly periodic sequences

In addition, the previous problems are equivalent to the following one in the case that $E(\mathbb{F}_q)$ is of odd order.

4. EDS Residue for non-perfectly periodic sequences

(perfectly periodic: period equal to order of point aka rank of apparition)

Elliptic nets

Connections

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Is this talk about combinatorics?

• combinatorial interpretations?

Connections

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Is this talk about combinatorics?

- combinatorial interpretations?
- Laurentness and positivity?

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- You tell me.