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Amicable pairs for elliptic curves

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joint work-in-progress with

Joseph H. Silverman Brown University / Microsoft Research

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Amicable Pairs

Definition

Let *E* be an elliptic curve defined over \mathbb{Q} . A pair (p, q) of primes is called an *amicable pair* for *E* if

$$#E(\mathbb{F}_p) = q$$
, and $#E(\mathbb{F}_q) = p$.

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$$#E(\mathbb{F}_p) = q$$
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Example

 $y^2 + y = x^3 - x$ has one amicable pair with $p, q < 10^7$:

(1622311, 1622471)

 $y^2 + y = x^3 + x^2$ has four amicable pairs with $p, q < 10^7$:

(853, 883), (77761, 77999), (1147339, 1148359), (1447429, 1447561).



Question (1) Let

 $Q_E(X) = \# \{ amicable pairs (p, q) such that p, q < X \}$

How does $Q_E(X)$ grow with X?



Questions

Question (1) Let

 $Q_E(X) = \# \{ amicable pairs (p, q) such that p, q < X \}$

How does $Q_F(X)$ grow with X?

Question (2) Let

 $\mathcal{N}_{E}(X) = \#\{\text{primes } p \leq X \text{ such that } \#E(\mathbb{F}_{p}) \text{ is prime}\}$

What about $\mathcal{Q}_{F}(X)/\mathcal{N}_{F}(X)$?

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 $\mathcal{N}_E(X)$

Let E/\mathbb{Q} be an elliptic curve, and let

 $\mathcal{N}_{E}(X) = \# \{ \text{primes } p \leq X \text{ such that } \# E(\mathbb{F}_{p}) \text{ is prime} \}.$

Conjecture (Koblitz, Zywina) There is a constant $C_{E/\mathbb{Q}}$ such that

$$\mathcal{N}_E(X) \sim C_{E/\mathbb{Q}} rac{X}{(\log X)^2}.$$

Further, $C_{E/\mathbb{Q}} > 0$ if and only if there are infinitely many primes p such that $\#E_p(\mathbb{F}_p)$ is prime.

 $C_{E/\mathbb{Q}}$ can be zero (e.g. if E/\mathbb{Q} has rational torsion).

Heuristic

Prob(*p* is part of an amicable pair)

 $= \operatorname{Prob}(q \stackrel{\text{def}}{=} \# E(\mathbb{F}_p) \text{ is prime}) \operatorname{Prob}(\# E(\mathbb{F}_q) = p).$



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$$\mathsf{Prob}(\# \mathsf{E}(\mathbb{F}_q) = p) \gg \ll rac{1}{\sqrt{q}} \sim rac{1}{\sqrt{p}}$$

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Together:

Prob(*p* is part of an amicable pair) $\gg \ll \frac{1}{\sqrt{p}(\log p)}$.

The CM case

Aliquot cycles

The j = 0 case

Final remarks

Growth of $Q_E(X)$

$$\mathcal{Q}_E(X) \approx \sum_{p \leq X} \operatorname{Prob}(p \text{ is part of an amicable pair })$$

 $\gg \ll \sum_{p \leq X} \frac{1}{\sqrt{p}(\log p)}$
 $\gg \ll \frac{\sqrt{X}}{(\log X)^2}.$

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Conjecture (Version 1)

Let E/\mathbb{Q} be an elliptic curve, let

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Assume infinitely many primes p such that $\#E(\mathbb{F}_p)$ is prime.



Conjectures

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Let E/\mathbb{Q} be an elliptic curve, let

 $Q_E(X) = \# \{ amicable pairs (p, q) such that p, q < X \}$

Assume infinitely many primes p such that $\#E(\mathbb{F}_p)$ is prime.

Then

$$\mathcal{Q}_E(X) \gg \ll rac{\sqrt{X}}{(\log X)^2}$$
 as $X o \infty$,

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where the implied constants depend on E.

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 $y^2 + y = x^3 + x^2$ has four amicable pairs with $p, q < 10^7$:

(853,883), (77761,77999), (1147339,1148359), (1447429,1447561).

 $y^2 = x^3 + 2$ has 5578 amicable pairs with $p, q < 10^7$:

 $(13, 19), (139, 163), (541, 571), (613, 661), (757, 787), \ldots$

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CM case: Twist Theorem

Theorem

Let E/\mathbb{Q} be an elliptic curve with complex multiplication by an order \mathcal{O} in a quadratic imaginary field $K = \mathbb{Q}(\sqrt{-D})$, with $j_E \neq 0$. Suppose that p and q are primes of good reduction for E with $p \ge 5$ and $q = \#E(\mathbb{F}_p)$.

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Then either

$$#E(\mathbb{F}_q) = \rho$$
 or $#E(\mathbb{F}_q) = 2q+2-\rho$.

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Then either

$$#E(\mathbb{F}_q) = p$$
 or $#E(\mathbb{F}_q) = 2q + 2 - p.$

In the latter case, $\#\tilde{E}(\mathbb{F}_q) = p$ for the non-trivial quadratic twist \tilde{E} of E over \mathbb{F}_q .

The j = 0 case

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Final remarks

CM case: Twist Theorem proof

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Eliminating curves with 2-torsion leaves $D \equiv 3 \mod 4$.



CM case: Twist Theorem proof

Eliminating curves with 2-torsion leaves $D \equiv 3 \mod 4$.

p splits as $p = \mathfrak{p}\overline{\mathfrak{p}}$ (if it were inert, we would have supersingular reduction, $\#E(\mathbb{F}_p) = p + 1$).



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 $#E(\mathbb{F}_p) = N(\Psi(\mathfrak{p})) + 1 - Tr(\Psi(\mathfrak{p}))$ where Ψ is the Grössencharacter of *E*.

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$$N(1 - \Psi(\mathfrak{p})) = \#E(\mathbb{F}_{\mathfrak{p}}) = \#E(\mathbb{F}_{p}) = q \text{ so } q \text{ splits as } q = q\overline{q}.$$

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 $N(1 - \Psi(\mathfrak{p})) = \#E(\mathbb{F}_{\mathfrak{p}}) = \#E(\mathbb{F}_{p}) = q \text{ so } q \text{ splits as } q = \mathfrak{q}\overline{\mathfrak{q}}.$ $N(\Psi(\mathfrak{q})) = q.$

So $1 - \Psi(\mathfrak{p}) = u\Psi(\mathfrak{q})$ for some unit $u \in \{\pm 1\}$.

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 $N(1 - \Psi(\mathfrak{p})) = \#E(\mathbb{F}_p) = \#E(\mathbb{F}_p) = q$ so q splits as $q = q\overline{q}$. $N(\Psi(q)) = q$. So $1 - \Psi(\mathfrak{p}) = u\Psi(q)$ for some unit $u \in \{\pm 1\}$.

 $Tr(\Psi(\mathfrak{q})) = \pm Tr(1 - \Psi(\mathfrak{p})) = \pm (2 - Tr(\Psi(\mathfrak{p}))) = \pm (q + 1 - p).$

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So $1-\Psi(\mathfrak{p})=u\Psi(\mathfrak{q}) ext{ for some unit } u\in\{\pm 1\}. \ &\mathcal{T}r(\Psi(\mathfrak{q}))=\pm\mathcal{T}r(1-\Psi(\mathfrak{p}))=\pm(2-\mathcal{T}r(\Psi(\mathfrak{p})))=\pm(q+1-p). \end{aligned}$
So...

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Twist frequencies for CM case

(<i>D</i> , <i>f</i>)	(3,3)	(11,1)	(19,1)	(43,1)	(67,1)	(163,1)
$X = 10^4$	18	8	17	42	48	66
$X = 10^5$	124	48	103	205	245	395
$X = 10^{6}$	804	303	709	1330	1671	2709
$X = 10^{7}$	5581	2267	5026	9353	12190	19691

Table: $Q_E(X)$ for elliptic curves with CM by $\mathbb{Q}(\sqrt{-D})$

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$X = 10^4$	0.217	0.250	0.233	0.300	0.247	0.237
$X = 10^{5}$	0.251	0.238	0.248	0.260	0.238	0.246
$X = 10^{6}$	0.250	0.247	0.253	0.255	0.245	0.247
$X = 10^{7}$	0.249	0.251	0.250	0.251	0.250	0.252

Table: $Q_E(X)/\mathcal{N}_E(X)$ for elliptic curves with CM by $\mathbb{Q}(\sqrt{-D})$

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Conjectures

Conjecture (Version 2)

Let E/\mathbb{Q} be an elliptic curve, let

 $\mathcal{Q}_{E}(X) = \# \{ amicable \ pairs (p,q) \ such \ that \ p,q < X \}$

Assume infinitely many primes p such that $\#E(\mathbb{F}_p)$ is prime.

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where the implied constants depend on E.

(b) If E has complex multiplication, then there is a constant $A_E > 0$ such that

$$\mathcal{Q}_E(X) \sim rac{1}{4} \mathcal{N}_E(X) \sim \mathcal{A}_E rac{X}{(\log X)^2}.$$

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Aliquot cycles

Definition

Let E/\mathbb{Q} be an elliptic curve. An *aliquot cycle of length* ℓ for E/\mathbb{Q} is a sequence of distinct primes $(p_1, p_2, \ldots, p_\ell)$ such that *E* has good reduction at every p_i and such that

$$\begin{split} \# E(\mathbb{F}_{\rho_1}) = \rho_2, \quad \# E(\mathbb{F}_{\rho_2}) = \rho_3, \quad \dots \\ \quad \# E(\mathbb{F}_{\rho_{\ell-1}}) = \rho_\ell, \quad \# E(\mathbb{F}_{\rho_\ell}) = \rho_1. \end{split}$$

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Example

$$y^2 = x^3 - 25x - 8$$
: (83, 79, 73)

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 $y^2 = x^3 + 176209333661915432764478x + 60625229794681596832262$:

(23, 31, 41, 47, 59, 67, 73, 79, 71, 61, 53, 43, 37, 29)

The j = 0 case

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Final remarks

No longer aliquot cycles in CM case

Theorem

A CM elliptic curve E/\mathbb{Q} with $j(E) \neq 0$ has no aliquot cycles of length $\ell \geq 3$ consisting of primes $p \geq 5$.

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Proof (sketch).

Postulate a cycle p_1, \ldots, p_ℓ (for a contradiction). Use CM theorem on pairs to write a linear recurrence relation for p_ℓ . See that it is strictly monotonic.

CM j = 0 case: Twist Theorem

$$\mathcal{K} = \mathbb{Q}(\sqrt{-3}), \qquad \mu_{\mathsf{6}} \subset \mathcal{O}_{\mathcal{K}} = \mathbb{Z}[\omega]$$



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Theorem

Let E/\mathbb{Q} be the elliptic curve $y^2 = x^3 + k$, and suppose that pand q are primes of good reduction for E with $p \ge 5$ and $q = \#E(\mathbb{F}_p)$. Then p splits in K, and we write $p\mathcal{O}_K = p\overline{p}$. Define $q = (1 - \Psi(p))\mathcal{O}_K$. Then we have $q\mathcal{O}_K = q\overline{q}$.

The values of the Grössencharacter at p and q are related by

$$1 - \Psi(\mathfrak{p}) = \left(\frac{4k}{\mathfrak{p}}\right)_6 \left(\frac{4k}{\mathfrak{q}}\right)_6 \Psi(\mathfrak{q}).$$

Finally, $\#E(\mathbb{F}_q) = p$ if and only if $\left(\frac{4k}{\mathfrak{p}}\right)_6 \left(\frac{4k}{\mathfrak{q}}\right)_6 = 1.$

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Data on twist frequencies

k	2	3	5	6	7	10
$X = 10^4$	0.217	0.141	0.097	0.085	0.165	0.118
$X = 10^5$	0.251	0.122	0.081	0.134	0.139	0.125
$X = 10^{6}$	0.250	0.139	0.083	0.142	0.133	0.107
$X = 10^{7}$	0.249	0.139	0.082	0.139	0.129	0.107

Table: $\mathcal{Q}_E(X)/\mathcal{N}_E(X)$ for elliptic curves $y^2 = x^3 + k$

 $1/12 = 0.08333\ldots$

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Applying Cubic Reciprocity

Let *E* be the curve $y^2 = x^3 + k$ and suppose $\#\tilde{E}_{\rho}(\mathbb{F}_{\rho})$ is prime.

$$\left(\frac{4k}{\Psi_E(\mathfrak{p})}\right)_6 \left(\frac{4k}{1-\Psi_E(\mathfrak{p})}\right)_6 = \cdots = \pm \left(\frac{\Psi_E(\mathfrak{p})(1-\Psi_E(\mathfrak{p}))}{k}\right)_3^{-1}$$

Applying Cubic Reciprocity

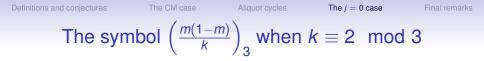
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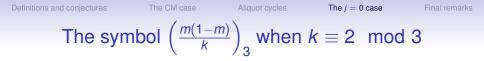
= ...
= $\pm \left(\frac{\Psi_E(\mathfrak{p})(1 - \Psi_E(\mathfrak{p}))}{k} \right)_3^{-1}$

Let M(k) be the number of elements in $\mathcal{O}_K/k\mathcal{O}_K$ for which m(1-m) is invertible. Let $M^*(k)$ be the number of those also satisfying $\left(\frac{m(1-m)}{k}\right)_3 = 1$. Then we may expect

$$\mathcal{Q}_E(X)/\mathcal{N}_E(X) \to M^*(k)/4M(k).$$

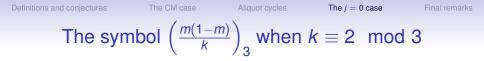






Then *E* is supersingular modulo *k* and has $(k + 1)^2$ points over $\mathbb{F}_{k\mathcal{O}_K} = \mathbb{F}_{k^2}$.

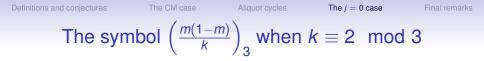
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Removing 3 points (∞ , (0,0) and (0,1)), the remaining points have $y \neq 0, 1$ and $\left(\frac{y(1-y)}{k}\right)_3 = 1$.

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Therefore, $((k + 1)^2 - 3)/3$ is the number of residues $m \neq 0, 1$ modulo $k\mathcal{O}_K$ having $\left(\frac{m(1-m)}{k}\right)_3 = 1.$

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Conjecture for j = 0 with k prime

$$\lim_{X\to\infty}\frac{\mathcal{Q}_k(X)}{\mathcal{N}_k(X)}=\frac{1}{6}+\frac{1}{2}R(k),$$

where R(k) depends on k (mod 36) and is given by:

k mod 36	R(k)	<i>k</i> mod 36	R(k)
1, 19	$\frac{2}{3(k-3)}$	17, 35	$\frac{2}{3(k-1)}$
13, 25	0	5, 29	0
7, 31	$\frac{2k}{3(k-2)^2}$	11, 23	$\frac{2k}{3(k^2-2)}$

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Data for j = 0 as k varies

					Density of Type I/II		
k	$\mathcal{Q}_k(X)$	$\mathcal{N}_k^{(1)}(X)$	$\mathcal{N}_k(X)$	$\mathcal{Q}/\mathcal{N}^{(1)}$	exper't	conjecture	
5 (b.2)	29340	58594	175703	0.251	0.3335	$\frac{1}{3} = 0.3333$	
7 (d.1)	43992	87825	168743	0.251	0.5205	$\frac{13}{25} = 0.5200$	
11 (d.2)	33721	66698	169062	0.253	0.3945	$\frac{47}{119} = 0.3950$	
13 (b.1)	28036	55766	167333	0.252	0.3333	$\frac{1}{3} = 0.3333$	
17 (a.2)	32008	63810	169226	0.251	0.3771	$\frac{3}{8} = 0.3750$	
19 (c.1)	31729	63066	168196	0.252	0.3750	$\frac{3}{8} = 0.3750$	
23 (d.2)	30480	61210	168512	0.249	0.3632	$\frac{191}{527} = 0.3624$	
29 (b.2)	28085	56286	168642	0.249	0.3338	$\frac{1}{3} = 0.3333$	
31 (d.1)	30301	60349	168344	0.251	0.3585	$\frac{301}{841} = 0.3579$	
37 (a.1)	29728	59430	168471	0.250	0.3528	$\frac{6}{17} = 0.3529$	
41 (b.2)	28050	56381	168567	0.249	0.3345	$\frac{1}{3} = 0.3333$	
43 (d.1)	29619	58807	168410	0.252	0.3492	$\frac{589}{1681} = 0.3504$	
47 (d.2)	29220	58400	168365	0.250	0.3469	$\frac{767}{2207} = 0.3475$	
53 (a.2)	29278	58257	168353	0.252	0.3460	$\frac{9}{26} = 0.3462$	
59 (d.2)	29378	58422	168783	0.252	0.3461	$\frac{1199}{3479} = 0.3446$	
61 (b.1)	28027	55816	168197	0.251	0.3318	$\frac{1}{3} = 0.3333$	
67 (d.1)	29242	57944	168239	0.253	0.3444	$\frac{1453}{4225} = 0.3439$	
71 (c.2)	28789	57661	168508	0.249	0.3422	$\frac{12}{35} = 0.3429$	

Table: Density of Amicable and Type I/II primes with $p \le X = 10^8$ for the curve $y^2 = x^3 + k$, prime *k*.

The j = 0 case

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Final remarks

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1. The predictions, even for the very complicated cases, are coming out to simple rational functions of k (all the point counting cancels). We don't have a simple explanation for this.

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2. One might look at this as a dynamical system: define a_n as in the L-series $L(E/\mathbb{Q}, s) = \sum_{n \ge 1} a_n/n^s$, and iterate the function $f(n) = n + 1 - a_n$ (future work).

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3. This question arises naturally from a question about when $n|W_n$ for an elliptic divisibility sequence (also work-in-progress). Smyth recently studied this for Lucas sequences.

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3. This question arises naturally from a question about when $n|W_n$ for an elliptic divisibility sequence (also work-in-progress). Smyth recently studied this for Lucas sequences.

4. We're currently running large searches to test the non-CM conjecture.