A dynamical system for elliptic divisibility sequences

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A Question

For any integer sequence $(D_n)_{n\geq 1}$ we define the *index divisibility set* of D to be

$$\mathcal{S}(D) = \{n \geq 1 : n \mid D_n\}.$$

Ex: $\mathcal{S}(D)$ for $D_n = b^n - b$ are pseudoprimes to the base b.

Strong divisibility sequences

Definition

An integer sequence $D_n, n \geq 1$ is a *divisibility sequence* if

$$n \mid m \implies D_n \mid D_m.$$

The sequence is a *strong divisibility sequence* if in addition

$$\gcd(D_n, D_m) \mid D_{\gcd(n,m)}.$$

Example (Fibonacci numbers)

n	1	2	3	4	5	6	7	8	9	10	11	12	13
F_n	1	1	2	3	5	8	13	21	34	55	89	144	233

Example (An elliptic divisibility sequence)

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
F_n	1	1	1	1	2	1	3	5	7	4	23	29	59	129

Rank of apparition

Definition

The *rank of apparition* of an integer $n \geq 1$ is

$$r_n = \min_{k>0}\{D_k \equiv 0 \pmod{n}\}$$

- The sequence r_n itself is a divisibility sequence: $n \mid m \implies r_n \mid r_m$.

Example (Fibonacci numbers)

n	1	2	3	4	5	6	7	8	9	10	11	12	13
F_n	1	1	2	3	5	8	13	21	34	55	89	144	233

Ranks of apparition:

n	1	2	3	4	5	6	7	8	9	10	11	12	13
r_n	1	3	4	6	5	12	8	6	12	15	10	12	7

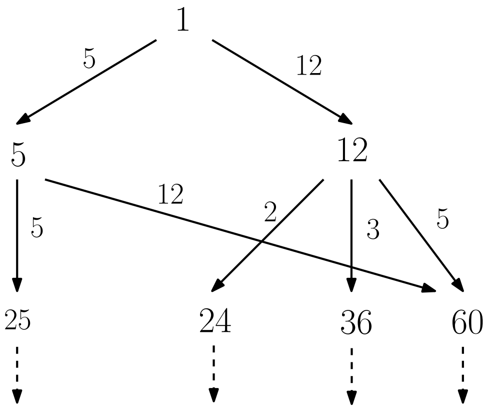
Index divisibility

Suppose $n \in \mathcal{S}(D)$, and p coprime to n satisfies $r_p = p$. Then $np \mid D_{np}$.

So if $n \in \mathcal{S}(D)$, then $np \in \mathcal{S}(D)$.

Make $\mathcal{S}(D)$ a directed graph with arrows $Arrow(D)$.

Index divisibility graph for Fibonacci numbers



n	1	2	3	4	5	6	7	8	9	10	11	12	13
F_n	1	1	2	3	5	8	13	21	34	55	89	144	233

A Theorem of Smyth

Theorem (Smyth)

Let $a, b \in \mathbb{Z}$, and let $L = (L_n)_{n \geq 1}$ be the associated Lucas sequence of the first kind, i.e.,

$$L_{n+2} = aL_{n+1} - bL_n, \qquad L_0 = 0, \quad L_1 = 1.$$

Let $\delta = a^2 - 4b$ and let $n \in \mathcal{S}(L)$ be a vertex. Then the arrows originating at n are

$$\{n \to np : p \text{ is prime and } p \mid L_n\delta\} \cup \mathcal{B}_{a,b,n},$$

where

$$\mathcal{B}_{a,b,n} = \begin{cases} \{n \to 6n\} & \text{if } (a,b) \equiv (3, \pm 1) \pmod 6, (6, L_n) = 1, \\ \{n \to 12n\} & \text{if } (a,b) \equiv (\pm 1, -1) \pmod 6, (6, L_n) = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

A dynamical system

Let D_n be a divisibility sequence. Define $\phi_D : \mathbb{Z}^{>0} \to \mathbb{Z}^{>0}$ be defined by

$$\phi_D(n) = r_n.$$

Example

Let F_n be the Fibonacci numbers. Then $\phi_F(5) = 5$ is the unique fixed point of ϕ_F.

Fibonacci numbers: Are there any 2-cycles consisting of prime numbers?

For the Fibonacci numbers F_n, and any prime number p, we have

$$r_p \mid p^2 - 1.$$

Suppose $\phi_F(p) = q$ and $\phi_F(q) = p$, for p, q odd primes. Then

$$q \mid (p+1)(p-1) \implies q \leq \frac{p+1}{2}$$

and similarly for p. So

$$q \leq \frac{p+1}{2}, \quad p \leq \frac{q+1}{2}.$$

So the answer is NO.

Elliptic divisibility sequences

Definition

Let $E/\mathbb{Q}$ be an elliptic curve and let $P \in E(\mathbb{Q})$ be a non-torsion point. The *elliptic divisibility sequence* (EDS) associated to the pair (E, P) is the sequence of positive integers D_n for $n \geq 1$ determined by

$$x\big([n]P\big) = \frac{A_n}{D_n^2} \in \mathbb{Q}$$

as a fraction in lowest terms.

Index divisibility in an EDS

Example

$$D_n : 1, 1, 1, 1, 2, 1, 3, 5, 7, 4, 23, 29, 59, 129,$$
$$314, 65, 1529, 3689, 8209, 16264, 83313, \ldots$$

$$E : y^2 + y = x^3 - x, \qquad P = (0, 0).$$

$$\mathcal{S}(D) = \{1, 40, 53, 63, 80, 127, 160, 189, 200, 320, 400, 441, 443, \ldots\}.$$

$$D_{40} = 40 \cdot 13526278251270010,$$
$$D_{53} = 53 \cdot 2997411336915768774003707557471.$$

Index divisibility for EDS

Theorem

Let D be a minimal regular EDS associated to the elliptic curve $E/\mathbb{Q}$ and point $P \in E(\mathbb{Q})$.

1. *If $n \in \mathcal{S}(D)$ and p is prime and $p \mid D_n$, then $(n \to np) \in \mathrm{Arrow}(D)$.*
2. *If $n \in \mathcal{S}(D)$ and d is an aliquot number for D and $\gcd(n, d) = 1$, then $(n \to nd) \in \mathrm{Arrow}(D)$.*
3. *If $p \geq 7$ is a prime of good reduction for E and if $(n \to np) \in \mathrm{Arrow}(D)$, then either $p \mid D_n$ or p is an aliquot number for D.*
4. *If $\gcd(n, d) = 1$ and if $(n \to nd) \in \mathrm{Arrow}(D)$ and if $d = p_1 p_2 \cdots p_\ell$ is a product of $\ell \geq 2$ distinct primes of good reduction for E satisfying $\min p_i > (2^{-1/2\ell} - 1)^{-2}$, then d is an aliquot number for D.*

Aliquot Number

Definition

Let D_n be an EDS, and let $p_1, \ldots, p_\ell$ be an ℓ-cycle for ϕ_D. That is,

$$p_{i+1} = r_{p_i} \qquad \text{for all } 1 \leq i \leq \ell,$$

(define $p_{\ell+1} = p_1$). Then $p_1 \cdots p_\ell$ is an *aliquot number*.

Fact

$p \mid D_n$ if and only if $[n]P = \mathcal{O} \pmod{p}$.

- So, if $\#E(\mathbb{F}_{p_i}) = p_{i+1}$ for each i, then the definition is satisfied.
- An anomalous prime ($\#E(\mathbb{F}_p) = p$) is an aliquot number.

Amicable Pairs

Definition

Let E be an elliptic curve defined over $\mathbb{Q}$. A pair (p, q) of primes is called an ***amicable pair*** for E if

$$\#E(\mathbb{F}_p) = q, \qquad \text{and} \qquad \#E(\mathbb{F}_q) = p.$$

Example

$y^2 + y = x^3 - x$ has one amicable pair with $p, q < 10^7$:

$$(1622311, 1622471)$$

$y^2 + y = x^3 + x^2$ has four amicable pairs with $p, q < 10^7$:

$$(853, 883), \quad (77761, 77999),$$
$$(1147339, 1148359), \quad (1447429, 1447561).$$

Aliquot cycles

Definition

An *aliquot cycle of length* ℓ for $E/\mathbb{Q}$ is a sequence of distinct primes $(p_1, p_2, \ldots, p_\ell)$ such that

$$\#E(\mathbb{F}_{p_1}) = p_2, \quad \#E(\mathbb{F}_{p_2}) = p_3, \quad \ldots \\ \#E(\mathbb{F}_{p_{\ell-1}}) = p_\ell, \quad \#E(\mathbb{F}_{p_\ell}) = p_1.$$

Example

$$y^2 = x^3 - 25x - 8 : (83, 79, 73)$$

$$E : y^2 = x^3 + 176209333661915432764478x + \\ 60625229794681596832262 :$$

$$(23, 31, 41, 47, 59, 67, 73, 79, 71, 61, 53, 43, 37, 29)$$

Constructing aliquot cycles with CRT

Let $p_1, p_2, \ldots, p_\ell$ be a sequence of primes such that

$$|p_i + 1 - p_{i+1}| \le 2\sqrt{p_i} \quad \text{for all } 1 \le i \le \ell,$$

(where $p_{\ell+1} = p_1$). For each p_i find (by Deuring) an elliptic curve $E_i/\mathbb{F}_{p_i}$ satisfying

$$\#E_i(\mathbb{F}_{p_i}) = p_{i+1}.$$

By the Chinese remainder theorem, find $E/\mathbb{Q}$ such that

$$E \bmod p_i \cong E_i \quad \text{for all } 1 \le i \le \ell.$$

Then $(p_1, \ldots, p_\ell)$ is an aliquot cycle of length ℓ for $E/\mathbb{Q}$.

A growth rate question

Question

Let

$$\mathcal{Q}_E(X) = \#\{\text{amicable pairs } (p, q) \text{ such that } p, q < X\}$$

How does $\mathcal{Q}_E(X)$ grow with X?

Heuristic

$$\begin{aligned}&\mathrm{Prob}(p \text{ is part of an amicable pair})\\ &\quad = \mathrm{Prob}(q \overset{\text{def}}{=} \#E(\mathbb{F}_p) \text{ is prime})\,\mathrm{Prob}(\#E(\mathbb{F}_q) = p).\end{aligned}$$

Conjecture of Koblitz:

$$\mathrm{Prob}(\#E(\mathbb{F}_p) \text{ is prime}) \asymp \frac{1}{\log p},$$

Conjecture of Sato–Tate:

$$\mathrm{Prob}(\#E(\mathbb{F}_q) = p) \asymp \frac{1}{\sqrt{q}} \sim \frac{1}{\sqrt{p}}.$$

Together:

$$\mathrm{Prob}(p \text{ is part of an amicable pair}) \asymp \frac{1}{\sqrt{p}(\log p)}.$$

$$\mathcal{Q}_E(X) \asymp \frac{\sqrt{X}}{(\log X)^2}$$

Conjectures

$$\mathcal{Q}_E(X) = \#\{\text{amicable pairs } (p,q) \text{ such that } p,q < X\}$$

Conjecture (Version 1)

Assume infinitely many primes p such that $\#E(\mathbb{F}_p)$ is prime.

Then

$$\mathcal{Q}_E(X) \asymp \frac{\sqrt{X}}{(\log X)^2} \quad \text{as } X \to \infty,$$

where the implied constants depend on E.

Unfortunately, Andrew Sutherland has only been able to find 117 amicable pairsless than 10^{12} on $y^2 + y = x^3 + x^2$.

Another example

$y^2 + y = x^3 - x$ has one amicable pair with $p, q < 10^7$:

$$(1622311, 1622471)$$

$y^2 + y = x^3 + x^2$ has four amicable pairs with $p, q < 10^7$:

$$(853, 883), \quad (77761, 77999),$$
$$(1147339, 1148359), \quad (1447429, 1447561).$$

$y^2 = x^3 + 2$ has 5578 amicable pairs with $p, q < 10^7$:

$$(13, 19), (139, 163), (541, 571), (613, 661), (757, 787), \ldots.$$

CM case: Twist Theorem

Theorem

Let $E/\mathbb{Q}$ be an elliptic curve ($j \neq 0$) with complex multiplication. Suppose that p and q are primes of good reduction for E with $p \geq 5$ and $q = \#E(\mathbb{F}_p)$.

Then either

$$\#E(\mathbb{F}_q) = p \qquad \text{or} \qquad \#E(\mathbb{F}_q) = 2q + 2 - p.$$

Remark: In the latter case, $\#\tilde{E}(\mathbb{F}_q) = p$ for the non-trivial quadratic twist $\tilde{E}$ of E over $\mathbb{F}_q$.

Pairs on CM curves

(D, f)	(3,3)	(11,1)	(19,1)	(43,1)	(67,1)	(163,1)
$X = 10^4$	18	8	17	42	48	66
$X = 10^5$	124	48	103	205	245	395
$X = 10^6$	804	303	709	1330	1671	2709
$X = 10^7$	5581	2267	5026	9353	12190	19691

Table: $\mathcal{Q}_E(X)$ for elliptic curves with CM

(D, f)	(3,3)	(11,1)	(19,1)	(43,1)	(67,1)	(163,1)
$X = 10^4$	0.217	0.250	0.233	0.300	0.247	0.237
$X = 10^5$	0.251	0.238	0.248	0.260	0.238	0.246
$X = 10^6$	0.250	0.247	0.253	0.255	0.245	0.247
$X = 10^7$	0.249	0.251	0.250	0.251	0.250	0.252

Table: $\mathcal{Q}_E(X)/\mathcal{N}_E(X)$ for elliptic curves with CM

Conjectures

$$\mathcal{Q}_E(X) = \#\{\text{amicable pairs } (p,q) \text{ such that } p,q < X\}$$

Conjecture (Version 2)

Assume infinitely many primes p such that $\#E(\mathbb{F}_p)$ is prime.

(a) If E does not have CM, then

$$\mathcal{Q}_E(X) \asymp \frac{\sqrt{X}}{(\log X)^2} \quad \text{as } X \to \infty,$$

where the implied constants depend on E.

(b) If E has CM, then there is a constant $A_E > 0$ such that

$$\mathcal{Q}_E(X) \sim A_E \frac{X}{(\log X)^2}.$$