Quantum computers and the future of cryptography

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Harvard University, Open Neighbourhood Seminar, October 9th, 2019

Classical and Quantum Computers

Classical Computers

- bit: boolean value
- gate: truth table
- space complexity: number of bits
- time complexity: number of sequential gates

Quantum Computers

- qubit: ?
- n entangled qubits: ?
- gate: ?
- space complexity: ?
- time complixity: ?

One qubit

A qubit is a complex linear combination of the two classical states $|0\rangle$ and $|1\rangle$:

$$a|0\rangle + b|1\rangle,$$

of length one:

$$a, b \in \mathbb{C}, \quad |a|^2 + |b|^2 = 1.$$

One qubit, measured

The state of a qubit is a complex linear combination of the two classical states $|0\rangle$ and $|1\rangle$:

$$a|0\rangle + b|1\rangle, \quad a, b \in \mathbb{C}, \quad |a|^2 + |b|^2 = 1.$$

We cannot know the state of the qubit. The one thing we can do is measure it:

- with probability $|a|^2$, it will become $|0\rangle$ and tell you "I'm 0!".
- with probability $|b|^2$, it will become $|1\rangle$ and tell you "I'm 1!".

Measuring destroys the state (it *collapses*) and replaces it with the answer, either $|0\rangle$ or $|1\rangle$.

You can never measure again.

Example

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

has equal probility of being measured as 0 or 1.

Two entangled qubits

The state of two (possibly entangled) qubits is a complex linear combination of all four possible classical states:

$$a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle,$$

of length one:

$$a, b, c, d \in \mathbb{C}, \quad |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1.$$

If we measure it, we obtain

- $|00\rangle$ with probability $|a|^2$
- $|01\rangle$ with probability $|b|^2$
- $|10\rangle$ with probability $|c|^2$
- $|11\rangle$ with probability $|d|^2$

Example

$$\frac{1}{\sqrt{4}}|00\rangle + \frac{1}{\sqrt{4}}|01\rangle + \frac{1}{\sqrt{4}}|10\rangle + \frac{1}{\sqrt{4}}|11\rangle$$

has equal probility of being measured as 00, 01, 10 or 11.

Example

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

has equal probility of being measured as 00, 11. It is not possible to measure 01 or 10.

Measuring one bit at a time

$$a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

It's also possible to measure just the second bit. In this case, we obtain

- 0 with probability $|a|^2 + |c|^2$ and the state collapses to $ak|00\rangle + ck|10\rangle$, where k is chosen to make $|ak|^2 + |ck|^2 = 1$
- 1 with probability $|b|^2 + |d|^2$ and the state collapses to $bk|01\rangle + dk|11\rangle$, where k is chosen to make $|bk|^2 + |dk|^2 = 1$

Example

$$\frac{1}{\sqrt{4}}|00\rangle + \frac{1}{\sqrt{4}}|01\rangle + \frac{1}{\sqrt{4}}|10\rangle + \frac{1}{\sqrt{4}}|11\rangle$$

If we measure the second qubit, we get

- 0 with probability $1/2$ and the state collapses to $1/\sqrt{2}|00\rangle + 1/\sqrt{2}|10\rangle$
- 1 with probability $1/2$ and the state collapses to $1/\sqrt{2}|01\rangle + 1/\sqrt{2}|11\rangle$

Suppose we are in the first case. Then if we continue on to measure the first qubit (after the second), we get 0 or 1 with equal probability.

Example

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

If we measure the second qubit, we get

- 0 with probability 1/2 and the state collapses to $|00\rangle$
- 1 with probability 1/2 and the state collapses to $|11\rangle$

Suppose we are in the first case. Then if measure the first qubit, we get 0 for sure.

This is weird, because if we'd measured it first, we would have gotten 0 or 1 with equal probability.

So measuring the second qubit forces the first to collapse, in some sense. This is *entanglement*.

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Quantum Computers

- qubit: unit vector in $\mathbb{C}^2$, i.e. a complex linear combination of the two states 0 and 1
- n entangled qubits: a linear combination of all 2^n possible classical bit states
- gate: ?
- space complexity: ?
- time complixity: ?

Quantum Gate

A *quantum gate* is a unitary matrix acting on the vector of coefficients of the state.

Quantum Gate

A *quantum gate* is a unitary matrix acting on the vector of coefficients of the state.
(*Unitary* just means it preserves the length of vectors.)

Example

The Hadamard gate is

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

It acts like this:

$$H(a|0\rangle + b|1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} a+b \\ a-b \end{pmatrix} = \frac{a+b}{\sqrt{2}}|0\rangle + \frac{a-b}{\sqrt{2}}|1\rangle.$$

For example, it takes $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ to $|0\rangle$.

Gates

A 2×2 quantum gate can act on a one-qubit state.
A 4×4 quantum gate can act on a two-qubit state.
An 8×8 quantum gate can act on a three-qubit state.
etc.

Example

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

This has the following effect:

$$|00\rangle \mapsto |00\rangle, \quad |01\rangle \mapsto |01\rangle, \quad |10\rangle \mapsto |11\rangle, \quad |11\rangle \mapsto |10\rangle.$$

One can think about the quantum gate via its action on basic states.
It is possible to perform standard arithmetic operations and normal logic on qubits, e.g.

$$|x, 0\rangle \mapsto |x, f(x)\rangle,$$

for functions $f(x)$ you might compute on a classical computer.

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- n entangled qubits: a linear combination of all 2^n possible classical bit states
- gate: unitary matrix
- space complexity: number of qubits
- time complixity: number of gates

Classical vs. Quantum Computation

Factoring an integer n:

- Classical: sub-exponential: $O\left(e^{C(\log n)^{1/3}(\log\log n)^{2/3}}\right)$ gates (General number field sieve)
- Quantum: polynomial: $O\left((\log n)^2(\log\log n)(\log\log\log n)\right)$ gates (Shor's algorithm)

Searching through n unordered items:

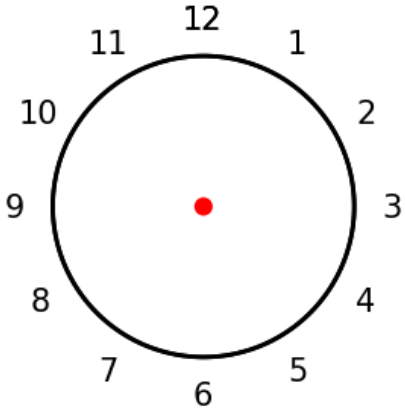
- Classical: exponential: $O(n)$ items tested
- Quantum: exponential: $O(\sqrt{n})$ items tested (Grover's Algorithm)

The power of a quantum computer

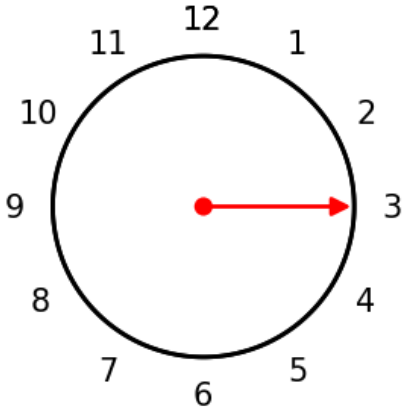
Power:
A quantum gate operates on *all the possible states simultaneously*.
(Think $|x, 0\rangle \mapsto |x, f(x)\rangle$.)

Limitation:
You can't see the result.

Interlude: modular arithmetic



Interlude: modular arithmetic



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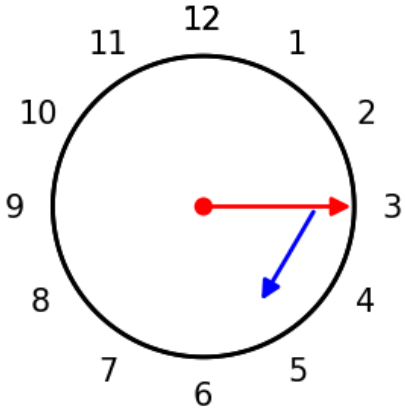

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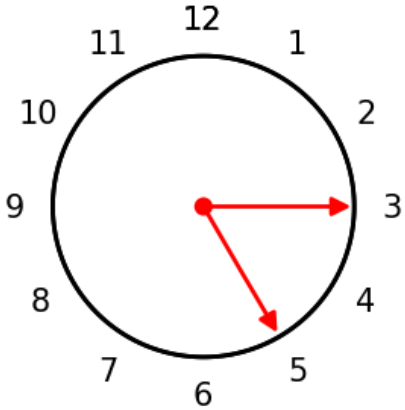

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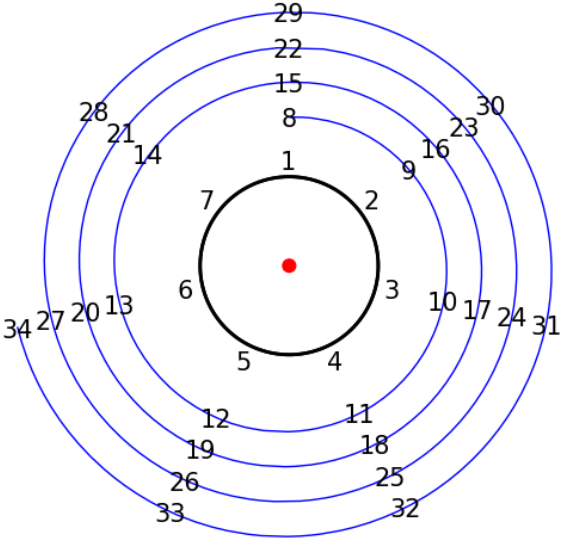


Interlude: modular arithmetic

Let n be an integer (the *modulus*). We can define

$$\mathbb{Z}/n\mathbb{Z}$$

to be the set of classes of integers mod n.

Two integers are in the same class if they differ by a multiple of n.

For example, modulo 3 there are three classes:

$$\begin{aligned} 0 &: \{\ldots, -3, 0, 3, 6, 9, \ldots\} \\ 1 &: \{\ldots, -2, 1, 4, 7, 10, \ldots\} \\ 2 &: \{\ldots, -1, 2, 5, 8, 11, \ldots\} \end{aligned}$$

Modular arithmetic: addition and multiplication

$$
\begin{aligned}
0 &: \{\ldots, -3, 0, 3, 6, 9, \ldots\} \\
1 &: \{\ldots, -2, 1, 4, 7, 10, \ldots\} \\
2 &: \{\ldots, -1, 2, 5, 8, 11, \ldots\}
\end{aligned}
$$

The amazing thing is that addition and multiplication respect this, e.g.

$$1+0, 1+6, 4+3, 7+0, \ldots$$

are all in the same class.

So we can define $\mathbb{Z}/n\mathbb{Z}$ to be n items (the classes), with an addition and multiplication law.

Example: $\mathbb{Z}/3\mathbb{Z}$

$$0 : \{\ldots, -3, 0, 3, 6, 9, \ldots\}$$
$$1 : \{\ldots, -2, 1, 4, 7, 10, \ldots\}$$
$$2 : \{\ldots, -1, 2, 5, 8, 11, \ldots\}$$

Addition

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Multiplication

	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Interlude: Fourier analysis

The space of functions

$$\{f : \{0, 1, 2, \ldots, N\} \to \mathbb{C}\}$$

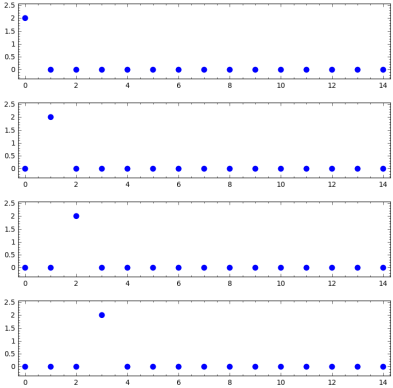
is a vector space.

We have addition and scalar multiplication.

Interlude: Fourier analysis

$$\{f : \{0, 1, 2, \ldots, N\} \to \mathbb{C}\}$$

Standard basis:



Interlude: Fourier analysis

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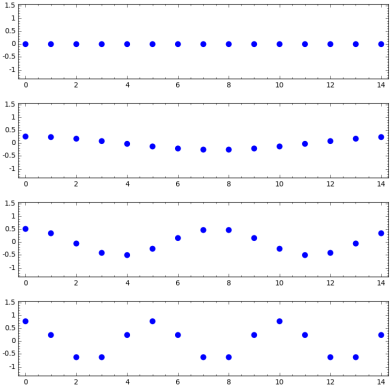
Fourier basis:



Interlude: Fourier analysis

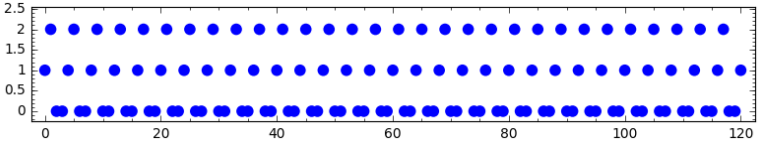
Discrete Fourier Transform:

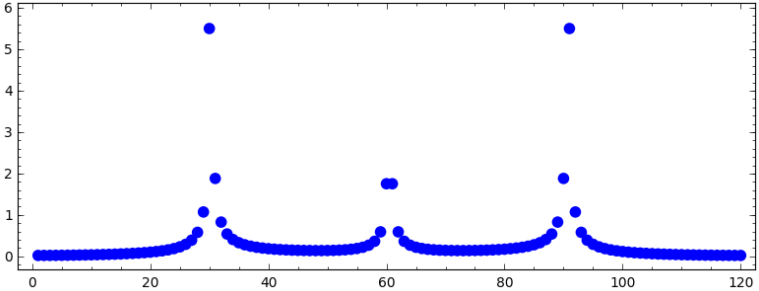
$$(x_j)_{j=1}^N \mapsto \left(\frac{1}{\sqrt{N}} \sum_{j=1}^N x_j e^{\frac{(2\pi i)jk}{N}} \right)_{k=1}^N$$

This changes between standard and Fourier bases.

Interlude: Fourier analysis

Discrete Fourier Transform detects frequency:
(N = 121, period = 4)





Quantum Fourier Transform

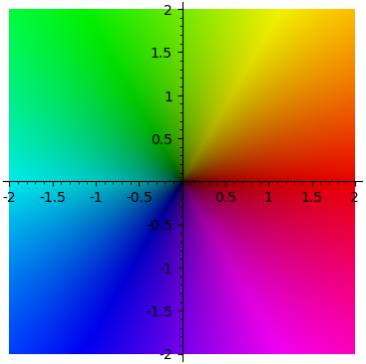
Given n qubits (2^n basic states), we define the QFT:

$$\sum_{j=1}^{N} x_j |j\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \left(\sum_{j=1}^{N} x_j e^{\frac{-(2\pi i)jk}{N}} \right) |k\rangle.$$

This is a unitary transformation on the vector of coefficients, i.e. a quantum gate.

QFT Demo

Domain coloring of the complex plane:



QFT Demo

DEMO

Lemma for Factoring on a Quantum Computer

In order to factor N, it is enough to know the period of the function

$$x \mapsto a^x \pmod{N}$$

for some a which is not too annoyingly trivial.

Hint: Fermat's Little Theorem says this order divides $\varphi(N)$.

Shor's algorithm for period finding

DEMO

How to factor N?

- classically reduce the problem to finding the period of a function $x \mapsto a^x \pmod N$.
- store a uniform linear combination of the states "x".
- use quantum gates to compute $x \mapsto a^x$ on all states simultaneously.
- result: a uniform linear combination of the states "$|x, a^x\rangle$".
- measure the second half: obtain a *periodic* linear combination of states.
- perform a *quantum fourier transform* to pick up the frequency.
- measure the result to read off the frequency.
- determine the period from the reading (classical).

Current public-key cryptography: the big three

- Factoring: Given $N = pq$, find p and q.
- Discrete Log mod p: Given $p, g, h = g^s \pmod{p}$, find s.
- Discrete Log on elliptic curves: Given E, P, $Q = [s]P$, find s.

Diffie-Hellman Key Exchange

Setup: p (modulus), g

Alice		Bob
Secret: a		Secret: b
$g^a \pmod p$	$\longrightarrow$	$g^a \pmod p$
$g^b \pmod p$	$\longleftarrow$	$g^b \pmod p$
Compute: $(g^b)^a \equiv g^{ab}$	$\leftarrow$ shared secret! $\rightarrow$	Compute: $(g^a)^b \equiv g^{ab}$

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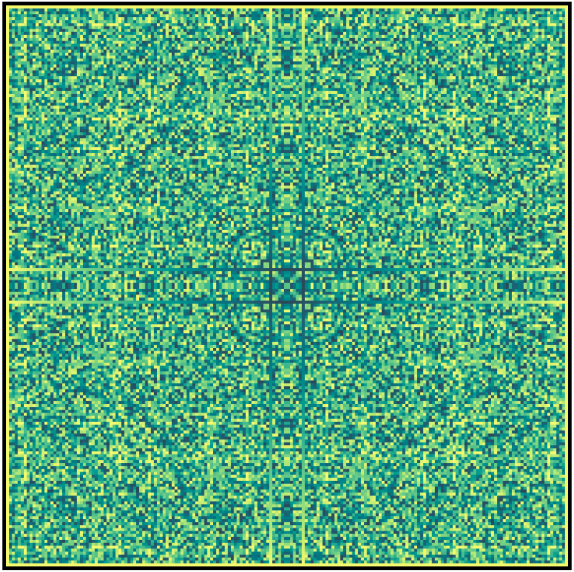
Hard Problem (Discrete Logarithm Problem)

Given $g, h = g^s \pmod p$, *find* s.

DLP Demo

DEMOS

Diffie-Hellman Carpets



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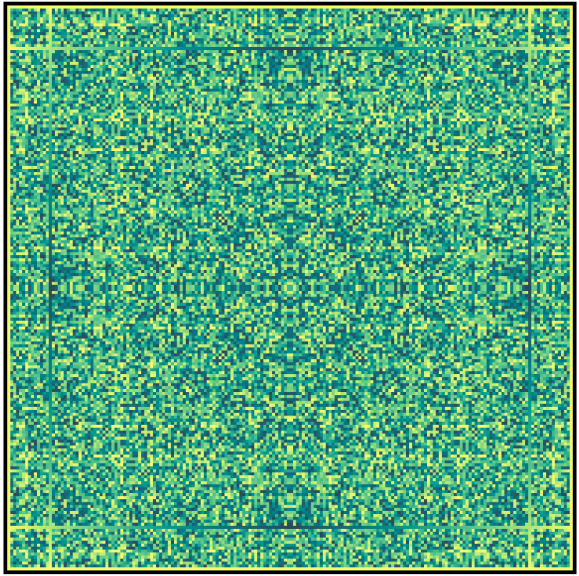
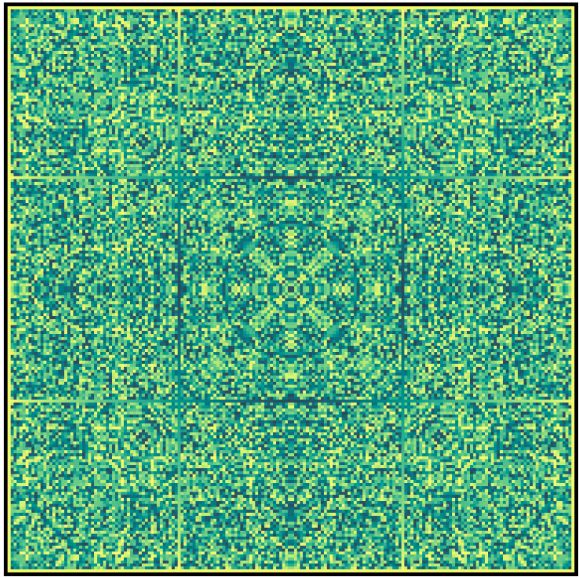


Diffie-Hellman Carpets



Adapting Shor's algorithm

In Shor's algorithm, the computer was able to find a *period* t such that $f(x) = f(x+t)$. This is an example of a *kernel* of a map.

Definition

The *kernel* of a map f is the set of inputs that give 0.

In particular, the set of periods is the kernel of the map

$$f : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}, \quad t \mapsto f(x+t) - f(x).$$

Shor's algorithm for discrete logarithm

Hard Problem (Discrete Logarithm Problem)

Given $g, h = g^s \pmod{p}$, *find* s.

We use a variant of Shor where we want to find the *kernel* of the function

$$f : \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}, \quad (a, b) \mapsto g^a h^{-b}.$$

The kernel of this function is exactly

$$\{(sx, x) : x\}$$

So if we can find any example of an element in this kernel, we can divide the two coordinates to obtain s.

Post-quantum candidates

- lattice-based cryptography
- isogeny-based cryptography
- code-based cryptography
- multivariate cryptography

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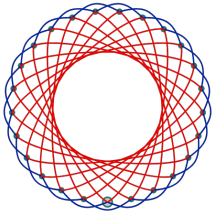
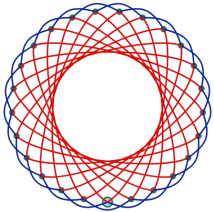
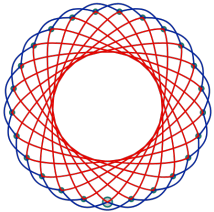
Isogeny-based cryptography: CSIDH

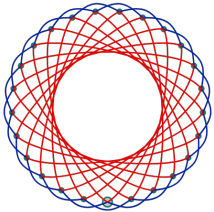
Alice





Bob





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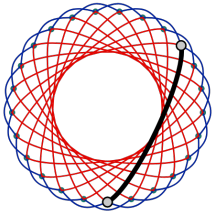

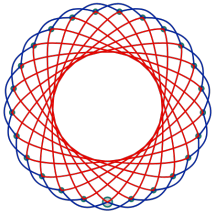

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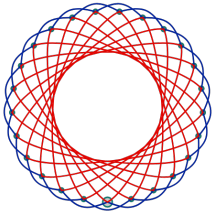

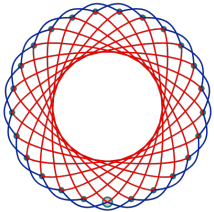

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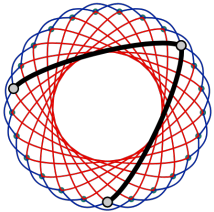

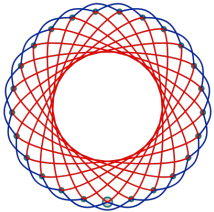

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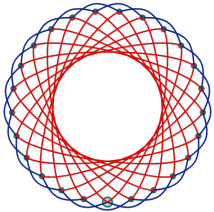

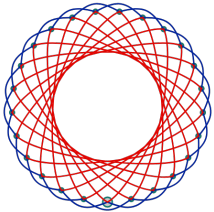

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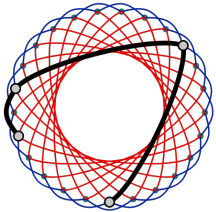

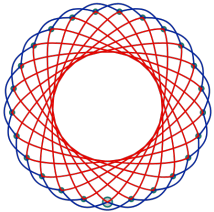

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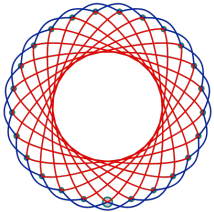

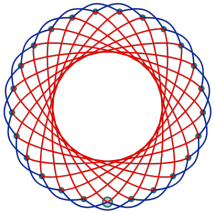

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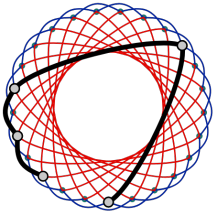

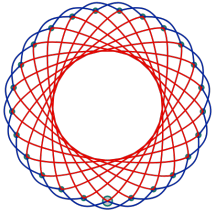

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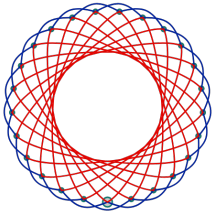

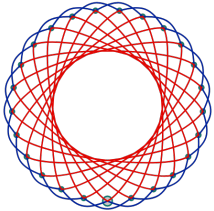

Isogeny-based cryptography: CSIDH

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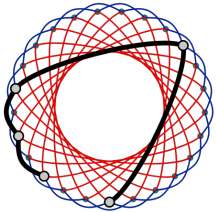

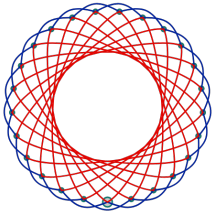

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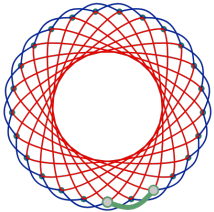

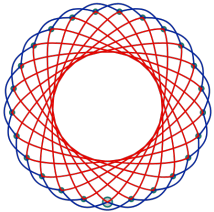

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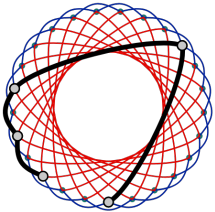

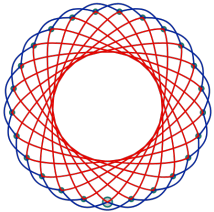

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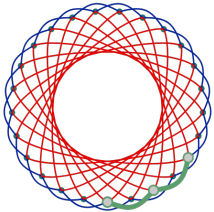

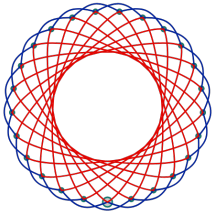

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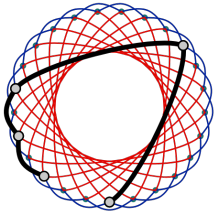

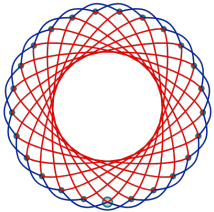

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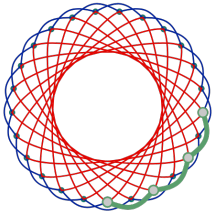

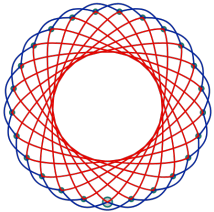

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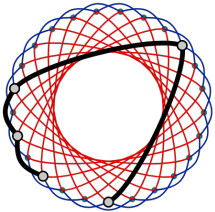

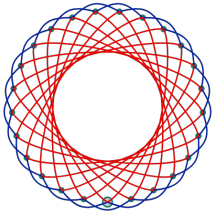

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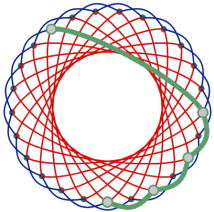

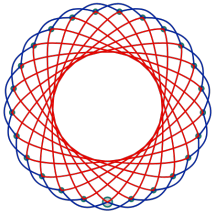

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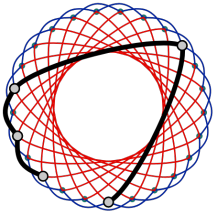

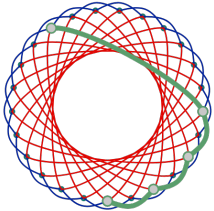

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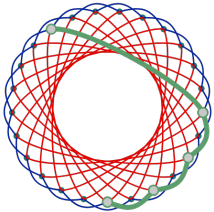

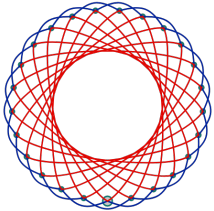

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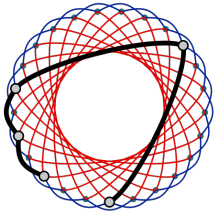

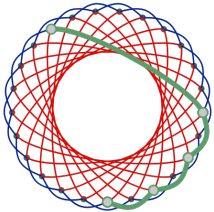

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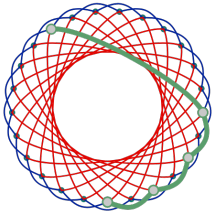

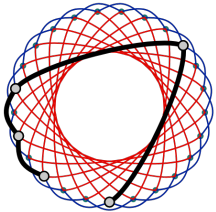

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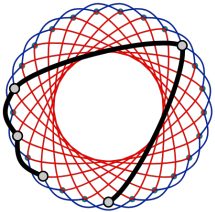

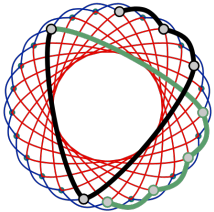

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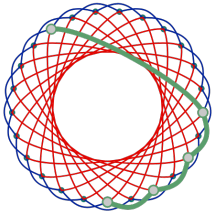

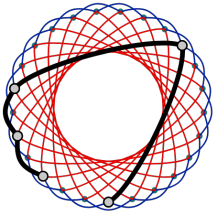

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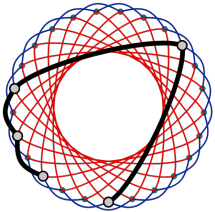

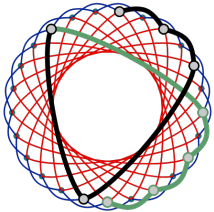

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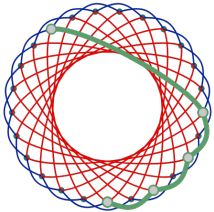

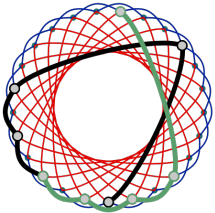

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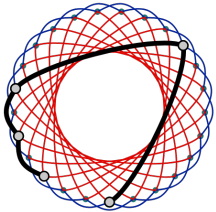

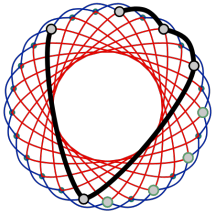

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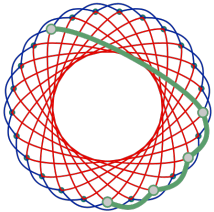
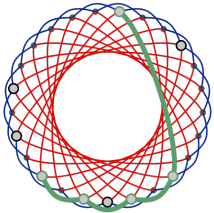


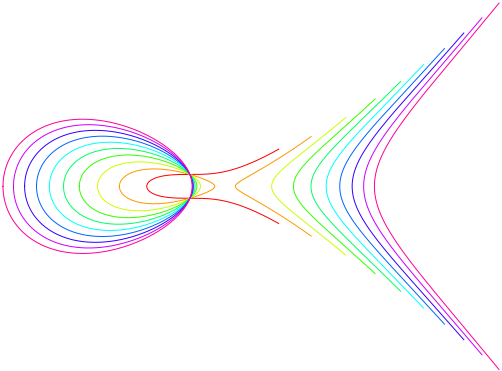


Elliptic Curves

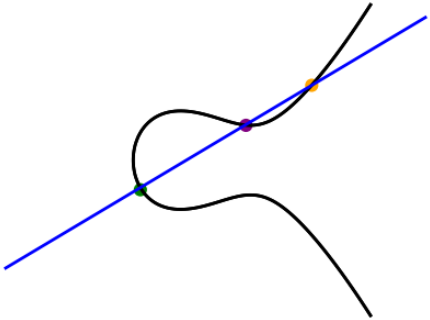
Definition

An *elliptic curve* is a smooth curve given by

$$y^2 = x^3 + ax + b$$



Addition law



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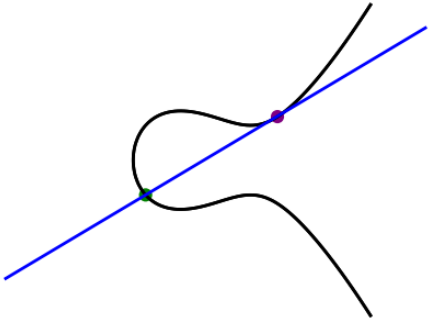

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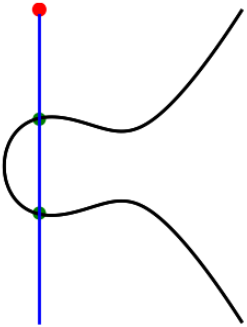

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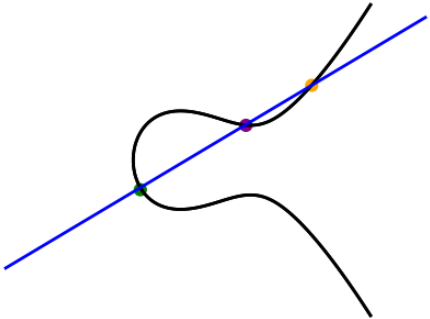

Three collinear points add to 0, the point at infinity.
Addition law is given by rational functions.

Addition law

The resulting group is *finite abelian group*, and the addition law is given by rational functions!
If $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$, then

$$P + Q = \left(\left(\frac{y_P - y_Q}{x_P - x_Q} \right)^2 - x_P - x_Q, -y_P - \left(\frac{(y_P - y_Q)(x_R - x_P)}{x_P - x_Q} \right) \right).$$

We can use this for an Elliptic Curve Discrete Logarithm Problem: given P and a multiple $[s]P$, find s.

Isogenies

A group homomorphism between elliptic curves is called an *isogeny* if it is realized as rational functions.

Example

Modulo 11, there is an isogeny from $y^2 = x^3 + 1$ to $y^2 = x^3 + 6$, given by

$$(x, y) \mapsto \left(\frac{x^3 + 4}{x^2}, \frac{x^3 y + 3y}{x^3} \right)$$

with kernel

$$\{(0, 1), (0, 10), id\}.$$

The *degree* of the isogeny is the cardinality of its kernel. (e.g. this is a 3-isogeny)

Isogeny graphs

$G_{\ell,n}$ mod q has

- vertices = elliptic curves mod q with n points (up to isomorphism)
  - label: $j = 1728\frac{4a^3}{4a^3+27b^2}$, isomorphism invariant
- edges = E and E' connected by an ℓ-isogeny
  - label: kernel of isogeny

For CSIDH, use the union of a few small ℓ, prime modulus p, and $n = p + 1$.

Hard Problem

Given two vertices (j-invariants) construct a path between them (sequence of isogenies/kernels).

Comparison to DLP

The graph has a *group action* (by a certain *class group*).

Hard Problem (Rephrased Isogeny Problem)

Given E and $s \cdot E$, compute s.

Hard Problem (Discrete Log Problem)

Modulo p, given g and g^s, compute s.

Shor's algorithm computes the 'period' (kernel) of the map

$$(x, y) \mapsto g^x (g^s)^{-y},$$

which gives away the secret s.

In our case, no way to 'multiply': $x \cdot E \overset{?}{=} -y \cdot (s \cdot E)$

Thank you!