Elliptic Nets

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Outline

Elliptic Divisibility Sequences
  Definitions
  Curve-Net Correspondence

Elliptic Nets
  Motivation
  Definitions
  Curve-Net Correspondence

Periodicity
  Elliptic Divisibility Sequences
  Elliptic Nets

Primitive Divisors
  Elliptic Divisibility Sequences
  Elliptic Nets
  Primes

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  - Definitions
  - Curve-Net Correspondence

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  - Elliptic Divisibility Sequences
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  - Primes

Definition and Examples

Definition

A sequence W is an *elliptic divisibility sequence* if for all positive integers $m > n$,

$$W_{m+n}W_{m-n}W_1^2 = W_{m+1}W_{m-1}W_n^2 - W_{n+1}W_{n-1}W_m^2 .$$

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- Example: $1, 1, -3, 11, 38, 249, -2357, 8767, 496036, -3769372, -299154043, -12064147359, \ldots$

Divisibility and Integrality

If $W_1, \ldots, W_4$ are integer with $W_1 = 1$, $W_2 W_3 \neq 0$, and $W_2 | W_4$, then the sequence ...

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3. if $gcd(W_3, W_4) = 1$, it satisfies the **Strong Divisibility Property**

$$W_{gcd(m,n)} = gcd(W_m, W_n) \text{ .}$$

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Ψ Functions

Let σ be the Weierstrass sigma function associated to the complex uniformization of an elliptic curve.

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$$\Psi_n(z) = \frac{\sigma(nz)}{\sigma(z)^{n^2}}$$

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- Elliptic functions.
- Simple zeroes at non-zero n-torsion points.
- Divisor is $\sum_{P \in E[n]} (P) - n^2(0)$.

Curve-Sequence Correspondence

Theorem (M. Ward, 1948)

Let E be an elliptic curve defined over $\mathbb{Q}$, and let $z \in \mathbb{C}$ correspond to a rational point P on E. Then

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- We call this the sequence associated to E, P.

Example: $y^2 + y = x^3 + x^2 - 2x$, $P = (0, 0)$

$$W_1 = 1$$
$$W_2 = 1$$
$$W_3 = -3$$
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$$W_5 = 38$$
$$W_6 = 249$$
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$$\begin{aligned}
W_1 &= 1 & P &= (0,0)\\
W_2 &= 1 & [2]P &= (3,5)\\
W_3 &= -3 & [3]P &= \left(-\frac{11}{9}, \frac{28}{27}\right)\\
W_4 &= 11 & [4]P &= \left(\frac{114}{121}, -\frac{267}{1331}\right)\\
W_5 &= 38 & [5]P &= \left(-\frac{2739}{1444}, -\frac{77033}{54872}\right)\\
W_6 &= 249 & [6]P &= \left(\frac{89566}{62001}, -\frac{31944320}{15438249}\right)\\
W_7 &= -2357 & [7]P &= \left(-\frac{2182983}{5555449}, -\frac{20464084173}{13094193293}\right)
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\end{aligned}$$

Division Polynomials

Any elliptic curve E has a Weierstrass equation. Suppose

$$E : y^2 = x^3 + Ax + B \ .$$

The elliptic functions $\Psi_n(z; \Lambda)$ can be written as **Division Polynomials** in terms of x, y, A, B:

$$\begin{aligned}
\Psi_1 &= 1, \\
\Psi_2 &= 2y, \\
\Psi_3 &= 3x^4 + 6Ax^2 + 12Bx - A^2, \\
\Psi_4 &= 4y(x^6 + 5Ax^4 + 20Bx^3 - 5A^2x^2 - 4ABx - 8B^2 - A^3),
\end{aligned}$$

Coordinates of $[n]P$

Suppose we have

$$E : y^2 = x^3 + Ax + B \ .$$

Define

$$\phi_n = x\Psi_n^2 - \Psi_{n+1}\Psi_{n-1} \ ,$$

$$4y\omega_n = \Psi_{n+2}\Psi_{n-1}^2 - \Psi_{n-2}\Psi_{n+1}^2 \ .$$

Then we have

$$[n]P = \left(\frac{\phi_n(P)}{\Psi_n(P)^2}, \frac{\omega_n(P)}{\Psi_n(P)^3} \right) \ .$$

Divisibility as a Curve Property I

1. We have the **Identity Property**:

$$W_n = 0 \iff [n]P = 0 \ .$$

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3. The Identity Property holds on this new curve over $\mathbb{F}_p$. Therefore there is some **Rank of Apparition** r of p in the sequence W_n such that

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4. This implies the divisibility property for squarefree numbers.

Divisibility as a Curve Property II

5. From the theory of formal groups, we also have the property that for r the rank of apparition of p:

$$v_p(W_{kr}) = v_p(W_r) + v_p(k)$$

Divisibility as a Curve Property II

5. From the theory of formal groups, we also have the property that for r the rank of apparition of p:

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6. Together with the last slide, this implies the **divisibility** of the elliptic divisibility sequence.

Singular Cases

In the case that one has a singular cubic curve

$$C : y^2 = x^3 + Ax + B$$

over $\mathbb{Q}$ and point $P \in C(\mathbb{Q})$, one can still consider the sequence of division polynomials $\Psi_n(P)$.

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Example (Singular Elliptic Divisibility Sequences)

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$$C : y^2 = x^3, P = (1, 1)$$

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$$C : y^2 = x^3 - \frac{25}{48}x + \frac{125}{864}, P = \left(\frac{17}{12}, \frac{3}{2}\right)$$

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Can we do more?

The elliptic divisibility sequence is associated to the sequence of points $[n]P$ on the curve.

$$\boxed{[n]P \leftrightarrow W_n}$$

The Mordell-Weil group of an elliptic curve may have rank > 1. We might dream of . . .

$$\boxed{[n]P + [m]Q \leftrightarrow W_{n,m}}$$

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Elliptic Nets

Definition (KS)

Let R be an integral domain, and A a finite-rank free abelian group. An *elliptic net* is a map $W : A \to R$ such that the following recurrence holds for all $p, q, r, s \in A$.

$$\begin{aligned} & W(p+q+s)W(p-q)W(r+s)W(r) \\ & \qquad + W(q+r+s)W(q-r)W(p+s)W(p) \\ & \qquad\qquad + W(r+p+s)W(r-p)W(q+s)W(q) = 0 \end{aligned}$$

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- The recurrence generates the net from finitely many initial values.
- The recurrence implies the elliptic divisibility sequence recurrence for $A = \mathbb{Z}$.

Example

$$E : y^2 + y = x^3 + x^2 - 2x; P = (0,0), Q = (1,0)$$

4335	5959	12016	-55287	23921	1587077	-7159461
94	479	919	-2591	13751	68428	424345
-31	53	-33	-350	493	6627	48191
-5	8	-19	-41	-151	989	-1466
1	3	-1	-13	-36	181	-1535
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Lattice Property

For an integer elliptic net, for each prime p, there exists a **Lattice of Apparition** $L \subset A$ such that

$$W(\mathbf{v}) \equiv 0 \mod p \iff \mathbf{v} \in L$$

The proof will wait until the curve-relationship is developed.

Scale Equivalence

- Let B, C be abelian groups. A quadratic function $f : B \to C$ is a function such that for all $x, y, z \in B$,

$$f(x+y+z)-f(x+y)-f(y+z)-f(x+z)+f(x)+f(y)+f(z)=0 \quad .$$

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  $$f(x+y+z)-f(x+y)-f(y+z)-f(x+z)+f(x)+f(y)+f(z)=0 \ .$$

- For any elliptic net $W : A \to K$, and quadratic $f : A \to K^*$, define $W^f : A \to K$ by

  $$W^f(\mathbf{v}) = f(\mathbf{v})W(\mathbf{v}) \ .$$

  This function is an elliptic net.

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$$W^f(\mathbf{v}) = f(\mathbf{v})W(\mathbf{v}) \quad .$$

  This function is an elliptic net.
- If two elliptic nets are related in the manner of W and W^f for some quadratic f, then we call them **Scale Equivalent.** This is clearly an equivalence relation.

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Elliptic Functions Ψ_n

Definition (M. Ward - Rank 1)

$$\Psi_n(z) = \frac{\sigma(nz)}{\sigma(z)^{n^2}}$$

- Elliptic functions.
- The function is zero if $nz = 0$.

Elliptic Functions $\Psi_{m,n}$

Definition (Rank 2)

$$\Psi_{n,m}(z,w) = \frac{\sigma(nz+mw)}{\sigma(z)^{n^2-nm}\sigma(z+w)^{nm}\sigma(w)^{m^2-nm}}$$

- Elliptic functions in each variable.
- The function is zero if $nz + mw = 0$.

Elliptic Functions $\Psi_{\mathbf{v}}$

Definition (Rank k)

$$\Psi_{\mathbf{v}}(\mathbf{z};\Lambda) = \frac{\sigma(v_1 z_1 + \ldots + v_k z_k;\Lambda)}{\prod_{1 \leq i \leq k} \sigma(z_i;\Lambda)^{2v_i^2 - \sum_{j=1}^{k} v_i v_j} \prod_{\substack{1 \leq i,j \leq k \\ i \neq j}} \sigma(z_i + z_j;\Lambda)^{v_i v_j}}$$

- Elliptic functions in each variable.
- The function is zero if $v_1 z_1 + \ldots + v_k z_k = 0$.

Elliptic Nets from Elliptic Curves

Theorem (KS)

Let E be an elliptic curve defined over $\mathbb{Q}$, and let $\mathbf{u} \in \mathbb{C}^k$ correspond to a vector of rational points $\mathbf{P} = (P_1, \ldots, P_k)$ on E. Then

$$W(\mathbf{v}) := \Psi_{\mathbf{v}}(\mathbf{u})$$

forms an elliptic net.

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$$W(\mathbf{v}) := \Psi_{\mathbf{v}}(\mathbf{u})$$

forms an elliptic net.

- We call this the elliptic net associated to the curve E and points $P_1, \ldots, P_k$.
- We call $P_1, \ldots, P_k$ the basis of the elliptic net.

Matrix and Homothety Actions on Elliptic Nets

Matrix Action:

- A $k \times l$ integer-coefficient matrix M acts on an elliptic net $W : \mathbb{Z}^k \to K$ by
$$W^M(\mathbf{v}) = W(M(\mathbf{v})) \ .$$

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Homothety Action:

- An element λ of K^* acts on an elliptic net $W : \mathbb{Z}^k \to K$ by
$$W^\lambda(\mathbf{v}) = \lambda W(\mathbf{v}) \ .$$

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- A $k \times l$ integer-coefficient matrix M takes E^k to E^l (integer-scalar multiplication and addition are defined via the curve group law).

Homothety Action:

- An element λ of K^* acts on an elliptic curve in Weierstrass form by the change of coordinates

$$(x, y) \mapsto (\lambda^2 x, \lambda^3 y) \ .$$

Singularity

One can define polynomials Δ and j in the values of an elliptic net. For elliptic divisibility sequences, these polynomials are

$$\begin{aligned}\Delta = (W_2^8 W_3^3)^{-1}(&-W_4^4 - 3W_2^5W_4^3 - 3W_2^{10}W_4^2 \\ &- 8W_2^2W_3^3W_4^2 - W_2^{15}W_4 + 20W_2^7W_3^3W_4 \\ &+ W_2^{12}W_3^3 - 16W_2^4W_3^6)\\ j = 64\Delta^{-1}(&W_2^{20} + 4W_4W_2^{15} - 16W_3^3W_2^{12} \\ &+ 6W_4^2W_2^{10} - 8W_4W_3^3W_2^7 + 4W_4^3W_2^5 + 16W_3^6W_2^4 \\ &+ 8W_4^2W_3^3W_2^2 + W_4^4)^3(W_3^4W_2^8)^{-3}\end{aligned}$$

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$$\begin{aligned}j = 64\Delta^{-1}(W_2^{20} + 4W_4 W_2^{15} - 16W_3^3 W_2^{12} \\ + 6W_4^2 W_2^{10} - 8W_4 W_3^3 W_2^7 + 4W_4^3 W_2^5 + 16W_3^6 W_2^4 \\ + 8W_4^2 W_3^3 W_2^2 + W_4^4)^3(W_3^4 W_2^8)^{-3}\end{aligned}$$

An elliptic net is called singular if $\Delta = 0$.

Curve-Net Correspondence

Theorem (KS)

Fix a field K. We have an explicit isomorphism of partially ordered sets

$$\left\{ \begin{array}{l} \textit{scale equivalence classes of non-singular elliptic nets} \\ W : \mathbb{Z}^k \to K \textit{ with } W(\mathbf{v}) \neq 0 \textit{ for } \mathbf{v} = \mathbf{e}_i, 2\mathbf{e}_i, 3\mathbf{e}_i \textit{ or } \mathbf{e}_i \pm \mathbf{e}_j \end{array} \right\}$$

$$\updownarrow$$

$$\left\{ \begin{array}{l} \textit{tuples } (E, \Omega, P_1, \ldots, P_k), \textit{ where } E \textit{ is an elliptic curve} \\ \textit{over } K, \Omega \textit{ is a holomorphic 1-form on } E \textit{ over } K, \\ P_i \in E(K) \backslash (E(K)[2] \cup E(K)[3]), \textit{ and } P_i \neq \pm P_j \textit{ for } i \neq j \end{array} \right\}$$

Furthermore, the matrix and homothety actions on the sets preserve the order and respect the isomorphism.

Lattice Property as a Curve Property

The elliptic net analogue of the Identity Property:

$$W(\mathbf{v}) \equiv 0 \mod p$$

$$\Longleftrightarrow$$

$$[v_1]P_1 + [v_2]P_2 + \cdots + [v_k]P_k = 0 \text{ on } E(\mathbb{F}_p)$$

This implies the Lattice Property.

Net Polynomials

Theorem (KS)

Suppose

$$f(x,y) = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6$$

gives an elliptic curve $E : f(x,y) = 0$*. The net functions* $\Psi_{\mathbf{v}}$ *on* E *can be expressed as polynomials in the ring*

$$\mathbb{Z}[a_1, a_2, a_3, a_4, a_6][x_i, y_i]_{i=1}^{k} \left[\frac{1}{x_i - x_j}\right]_{1\leq i<j\leq k} \Bigg/ \langle f(x_i, y_i)\rangle_{i=1}^{k} \quad .$$

Net Polynomial Examples

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\Psi_{1,1,1} &= \frac{y_1(x_2 - x_3) + y_2(x_3 - x_1) + y_3(x_1 - x_2)}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)} \ ,
\end{aligned}$$

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\Psi_{3,1} &= (4x_1^6 - 12x_2x_1^5 + 9x_2^2x_1^4 + 4x_2^3x_1^3 - 4y_2^2x_1^3 + 8y_1^2x_1^3 - 6x_2^4x_1^2 \\
&\quad + 6y_2^2x_2x_1^2 - 18y_1^2x_2x_1^2 + 12y_1^2x_2^2x_1 + x_2^6 - 2y_2^2x_2^3 - 2y_1^2x_2^3 \\
&\quad + y_2^4 - 6y_1^2y_2^2 + 8y_1^3y_2 - 3y_1^4)(x_2 - x_1)^{-3} \ .
\end{aligned}
$$

A Note About the Proofs I

1. **Curves give Nets over** $\mathbb{C}$: Check the recurrence – classical complex elliptic function theory.

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1. **Curves give Nets over** $\mathbb{C}$: Check the recurrence – classical complex elliptic function theory.
2. **Find a sufficiently simple baseset for nets under the recurrence**: Complicated nested inductions.
3. **Show the** Ψ **are polynomial of a nice form on the baseset**: Classical complex elliptic function theory.
4. **Extend these net polynomials of the baseset to any field**: Choose an appropriate fibration of E^n over an appropriate ring. Extend the divisors of the Ψ functions from the fibre over $\mathbb{Q}$ and check that there are no vertical components.

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5. **Show the Ψ are polynomial of a nice form in general**: Use the inductive function theory of Step 2 to show that this type of extension can be done in general.

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7. **Nets give Curves in Rank 1 and 2**: Explicitly calculate the relevant curve and check agreement on the baseset, which implies agreement everywhere.
8. **Nets give Curves in All Ranks**: Induction from the base case of ranks 1 and 2 by considering subnets.

Outline

- Elliptic Divisibility Sequences
  - Definitions
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Ward's Periodicity Property

If P is an r-torsion point, W is the elliptic net associated to E, P, then

$$W(r+k) \text{ is not necessarily equal to } W(k) \quad .$$

Example

$E : y^2 + y = x^3 + x^2 - 2x$ over $\mathbb{F}_5$.
$P = (0,0)$ has order 9.
The associated sequence is
$0, 1, 1, 2, 1, 3, 4, 3, 2, 0, 3, 2, 1, 2, 4, 3, 4, 4, 0, 1, 1, 2, 1, 3, 4, \ldots$

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Periodicity for Elliptic Divisibility Sequences

Theorem (M. Ward, 1948)

Let W *be an elliptic divisibility sequence, and* $p \geq 3$ *a prime not dividing* $W(2)W(3)$*. Let* r *be the least positive integer such that* $W(r) \equiv 0 \mod p$*. Then there exist integers* a, b *such that for all* n*,*

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Example ($E : y^2 + y = x^3 + x^2 - 2x$, $P = (0, 0)$ over $\mathbb{F}_5$)

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$k = 2 : W(18 + n) \equiv W(n)4^{2n}2^4 \equiv W(n) \mod 5$

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Reducing a Net Modulo p

Corollary (KS - Corollory to Curve-Net Theorem)

Let E be an elliptic curve over K and let $P_1, \ldots, P_k$ be K-points of E. Let $\widetilde{E}$ and $\widetilde{P}_1, \ldots, \widetilde{P}_k$ be their reductions modulo a prime p. Then the elliptic net associated to $\widetilde{E}, \widetilde{P}_1, \ldots, \widetilde{P}_k$ is the reduction modulo p of the elliptic net associated to $E, P_1, \ldots, P_k$.

Example of Reduction Mod 5 of an Elliptic Net

0	4	4	3	1	2	4
4	4	4	4	1	3	0
4	3	2	0	3	2	1
0	3	1	4	4	4	4
1	3	4	2	4	1	0
1	1	2	0	2	4	1
0	1	1	2	1	3	4

$\uparrow$ Q

$P \longrightarrow$

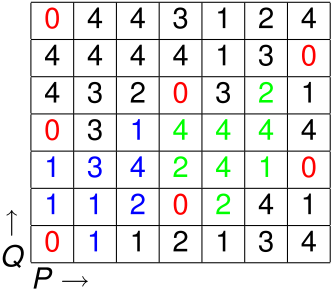
Example of Reduction Mod 5 of an Elliptic Net

0	4	4	3	1	2	4
4	4	4	4	1	3	0
4	3	2	0	3	2	1
0	3	1	4	4	4	4
1	3	4	2	4	1	0
1	1	2	0	2	4	1
0	1	1	2	1	3	4

$\uparrow$ Q

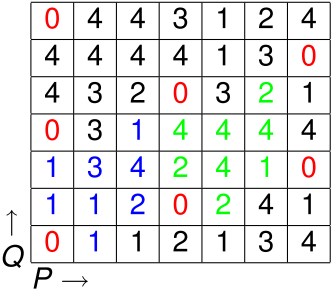
$P \longrightarrow$

Example of Reduction Mod 5 of an Elliptic Net



0	4	4	3	1	2	4
4	4	4	4	1	3	0
4	3	2	0	3	2	1
0	3	1	4	4	4	4
1	3	4	2	4	1	0
1	1	2	0	2	4	1
0	1	1	2	1	3	4

Example of Reduction Mod 5 of an Elliptic Net



0	4	4	3	1	2	4
4	4	4	4	1	3	0
4	3	2	0	3	2	1
0	3	1	4	4	4	4
1	3	4	2	4	1	0
1	1	2	0	2	4	1
0	1	1	2	1	3	4

The appropriate periodicity property should tell how to obtain the green values from the blue values.

Periodicity for Elliptic Nets

Theorem (KS)

Let $W : \mathbb{Z}^2 \to K$ *be an elliptic net such that* $W(2,0)W(0,2) \neq 0$*.*
Suppose $W(r_1, r_2) = W(s_1, s_2) = 0$*. Then there exist*
$a_s, b_s, c_s, a_r, b_r, c_r, d \in K$ *such that for all* $m, n, k, l \in \mathbb{Z}$*,*

$$W(kr_1 + ls_1 + m, kr_2 + ls_2 + n) = W(m,n)a_r^{km}b_r^{kn}c_r^{k^2}a_s^{lm}b_s^{ln}c_s^{l^2}d^{kl}$$

In particular, if K is a finite field such as $\mathbb{F}_p$, we obtain a statement about reduction modulo p (i.e. for an integer elliptic net, if $W(r_1, r_2)$ and $W(s_1, s_2)$ are trivial mod p, then the equation holds mod p).

Example of Net Periodicity

0	4	4	3	1	2	4
4	4	4	4	1	3	0
4	3	2	0	3	2	1
0	3	1	4	4	4	4
1	3	4	2	4	1	0
1	1	2	0	2	4	1
0	1	1	2	1	3	4

↑ Q

P →

Example of Net Periodicity

0	4	4	3	1	2	4
4	4	4	4	1	3	0
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0	3	1	4	4	4	4
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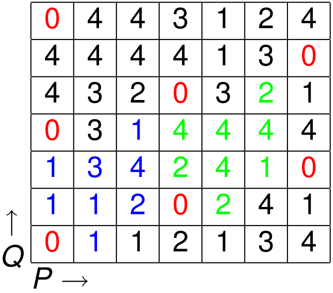
1	1	2	0	2	4	1
0	1	1	2	1	3	4

$\uparrow$ Q

$P \rightarrow$

Example of Net Periodicity



0	4	4	3	1	2	4
4	4	4	4	1	3	0
4	3	2	0	3	2	1
0	3	1	4	4	4	4
1	3	4	2	4	1	0
1	1	2	0	2	4	1
0	1	1	2	1	3	4

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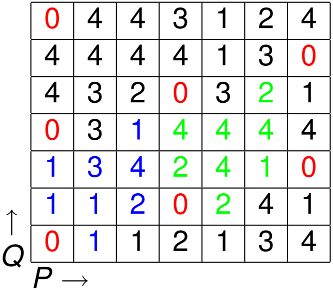

0	4	4	3	1	2	4
4	4	4	4	1	3	0
4	3	2	0	3	2	1
0	3	1	4	4	4	4
1	3	4	2	4	1	0
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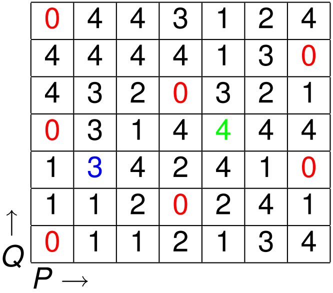
$$a_r = 2, b_r = 2, c_r = 1$$

Example of Net Periodicity



0	4	4	3	1	2	4
4	4	4	4	1	3	0
4	3	2	0	3	2	1
0	3	1	4	4	4	4
1	3	4	2	4	1	0
1	1	2	0	2	4	1
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$$a_r = 2, b_r = 2, c_r = 1$$

$$W(5,4) \equiv W(1,2)2^1 2^2 1^1$$
$$\equiv 3W(1,2) \mod 5$$

Periodicity from Pairings

For those that know the Tate and Weil pairings, the periodicity contains the values of these pairings. In the previous theorems,

- $a = T_r(P, P)$
- a_r, a_s, b_r, b_s are appropriate Tate pairings of multiples of P and Q

The Tate and Weil pairings can therefore be calculated from elliptic nets efficiently. This is of interest to pairing-based elliptic-curve cryptographers.

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Primitive Divisors in Elliptic Divisibility Sequences

We may define a **Primitive Divisor** of a term W_n to be a prime p such that $p | W_n$ and $p \not| W_m$ for any $0 < m < n$. We then have

Primitive Divisors in Elliptic Divisibility Sequences

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Theorem (Silverman's Elliptic Zsigmondy Theorem)

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There have since been many other results...

Outline

Elliptic Divisibility Sequences
  - Definitions
  - Curve-Net Correspondence

Elliptic Nets
  - Motivation
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Periodicity
  - Elliptic Divisibility Sequences
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Primitive Divisors
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  - Elliptic Nets
  - Primes

The Canonical Height

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This is a quadratic form with an associated bilinear form $\langle \cdot, \cdot \rangle$. For us, what's relevant is that

$$\log |D_n| \sim \hat{h}(P) n^2 \quad ,$$

and in fact for elliptic nets

$$\log |W_{\mathbf{v}}| \sim \hat{h}(\mathbf{v} \cdot \mathbf{P}) = \sum_{i,j=1}^{k} v_i v_j \left\langle P_i, P_j \right\rangle \quad .$$

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Possible Definition

Let p be a primitive divisor for a term $W_{\mathbf{v}}$ if

$$\hat{h}(\mathbf{v}\cdot\mathbf{P}) = \min\left\{\hat{h}(\mathbf{u}\cdot\mathbf{P}) \text{ such that } p|W_{\mathbf{u}}, \mathbf{u}\neq 0\right\} \quad .$$

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Geometrically, "for all points P of sufficient height, is it true that P is the point of least height in some kernel of reduction?"

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- Proofs??

For Further Reading I

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