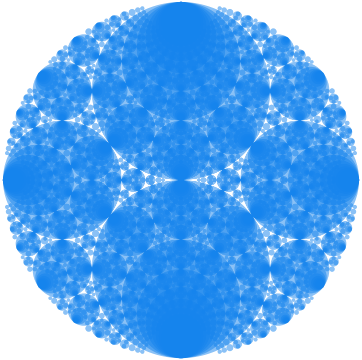
Illustrating the arithmetic of imaginary quadratic fields

Katherine E. Stange

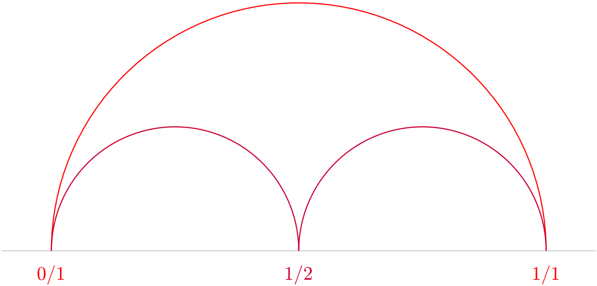


Colloquium, Montana State University, November 4, 2019

the Farey subdivision

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

the Farey subdivision



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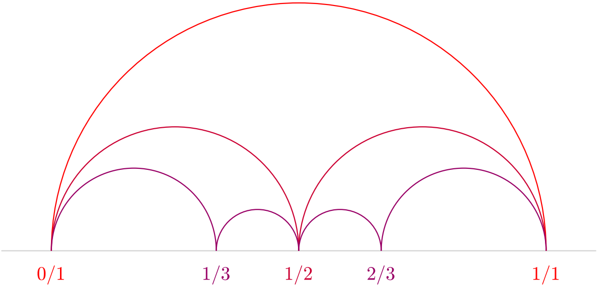

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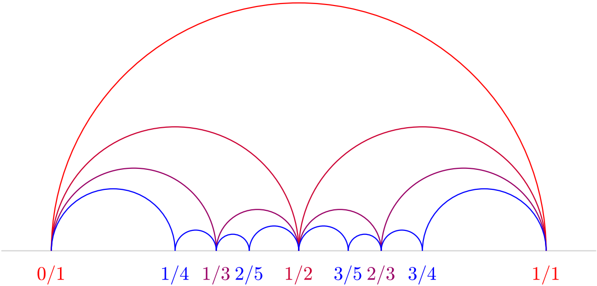

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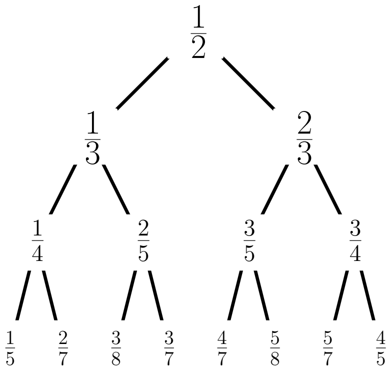
An arborist's view of $\mathbb{P}^1(\mathbb{Z})$



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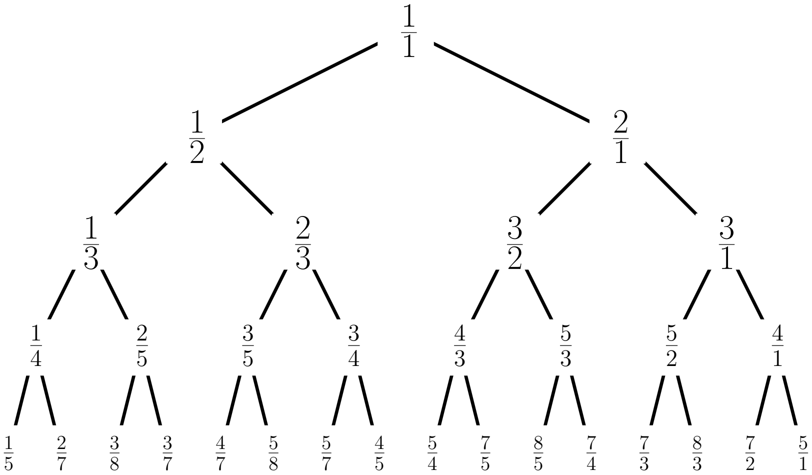
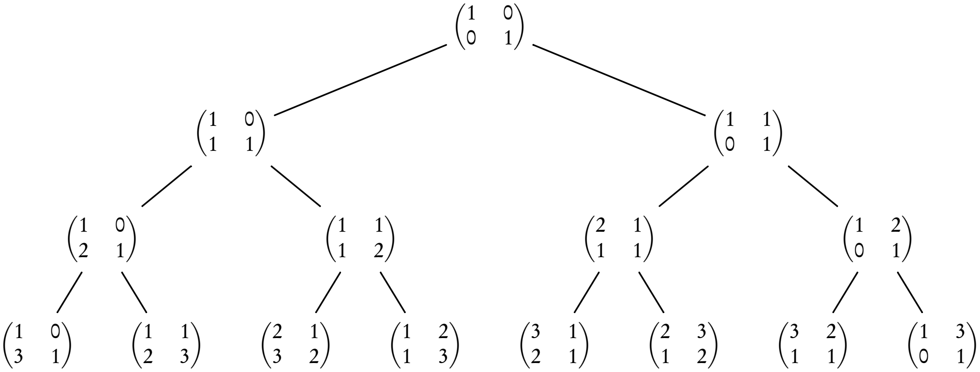
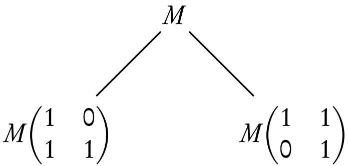


Money may not, but matrices do: $SL_2^+(\mathbb{Z})$





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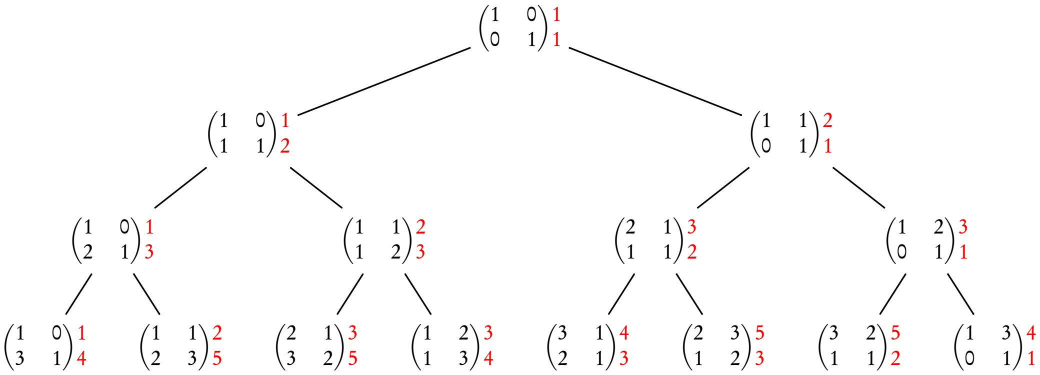

$$M \mapsto M\begin{pmatrix}1\\1\end{pmatrix}$$

Diophantine approximation: the address of $\alpha \in \mathbb{R}$

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

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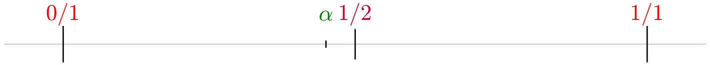

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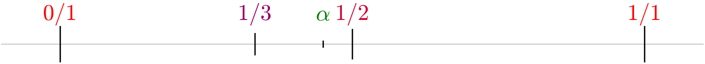

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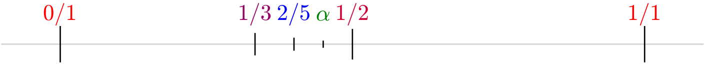

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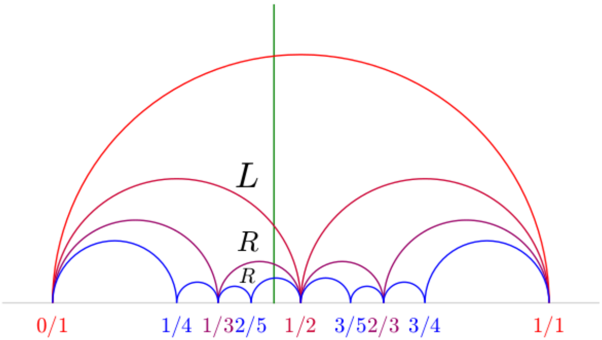
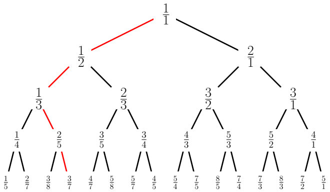


Diophantine approximation: the address of $\alpha \in \mathbb{R}$

Real number α



Infinite path through tree:

$$L^{a_0} R^{a_1} L^{a_2} R^{a_3} \cdots$$

Matrix product:

$$\begin{pmatrix} 1 & 0 \\ a_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_3 \\ 0 & 1 \end{pmatrix} \cdots$$

Farey tesselation of the upper half plane

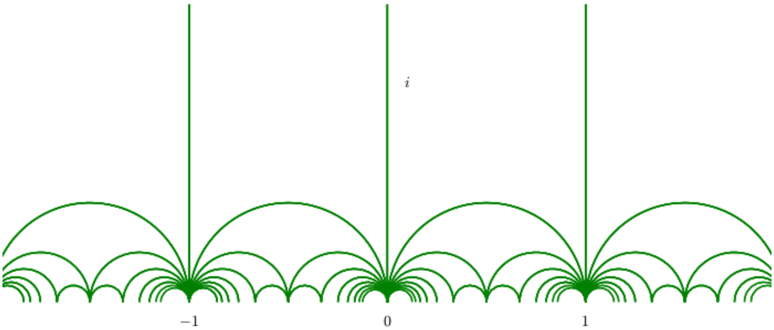
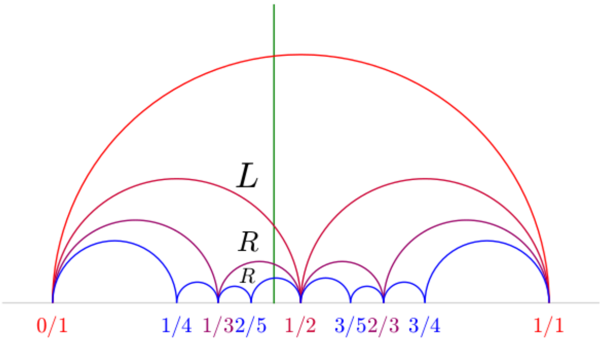


Image of $\{0, \infty\}$ (and its hyperbolic geodesic) under $PSL_2(\mathbb{Z})$ action:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \mapsto \quad \left(z \mapsto \frac{az+b}{cz+d} \right)$$

Geodesic viewpoint



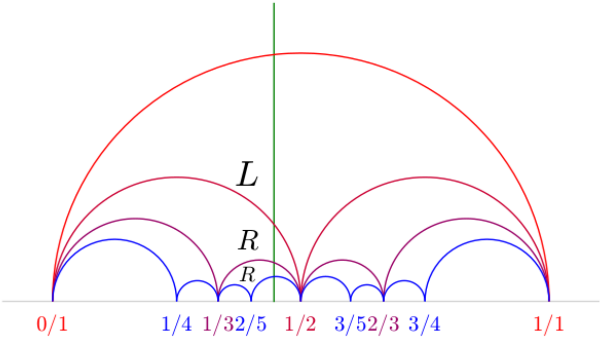
The Farey subdivision: Continued fractions / Euclidean algorithm

$$p_n/q_n = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \frac{1}{a_n}}}}}$$

$$\begin{aligned} x_{n+2} &= -x_{n+1}a_n + x_n \\ &\vdots \\ x_4 &= -x_3 a_2 + x_2 \\ x_3 &= -x_2 a_1 + x_1 \\ x_2 &= -x_1 a_0 + x_0 \end{aligned}$$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_3 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & a_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The Farey endpoints



endpoints of pierced bubbles are good approximations:

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

Diophantine approximation and continued fractions

Question: For a given $\alpha \in \mathbb{R}$, when does

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

have finitely/infinitely many solutions p/q?

Answer (Dirichlet): infinitely many if and only if α is irrational.

Theorem

The convergents p_n/q_n given by the continued fraction algorithm are the best approximations in the sense of:

- $|z - p_n/q_n| < 1/|a_n q_n^2|$
- $|q_n z - p_n| < \varepsilon |q_{n-1} z - p_{n-1}|$
- $|q_n| > 1/\varepsilon^n$
- *If $p, q \in \mathbb{Z}$ with $|q| < c''|q_n|$, then $|q_n z - p_n| < c'|qz - p|$.*

Diophantine approximation: algebraic numbers are poorly approximable

Question 1: For a given $\alpha \in \mathbb{R}$, when does

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

have finitely/infinitely many solutions p/q?

Answer (Dirichlet): infinitely many if and only if α is irrational.

Question 2: What if we ask for $< \frac{1}{q^{2+\epsilon}}$?

Answer (Roth): if α is algebraic, only finitely many.

Can we approximate complex numbers?

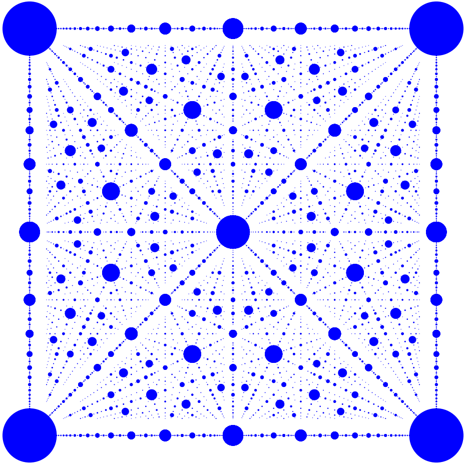
Perhaps with Gaussian rationals?

$$\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$$
$$\mathbb{Q}(i) = \{a + bi : a, b \in \mathbb{Q}\}$$

Rationals $\mathbb{Q}$

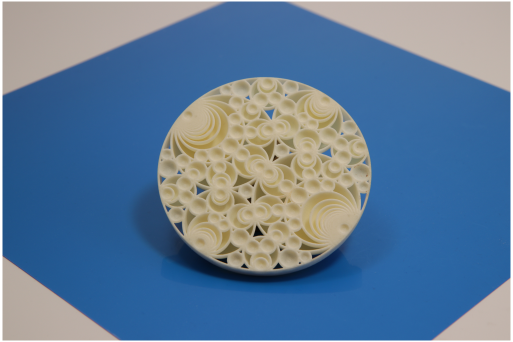
sized by norm of the denominator

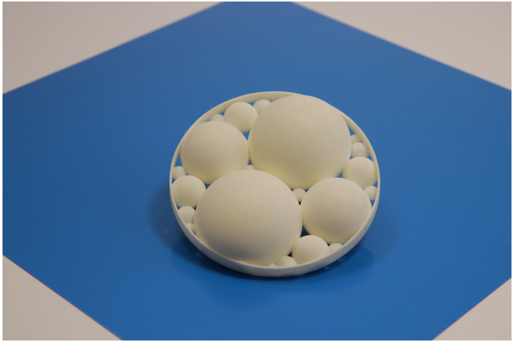
Gaussian rationals



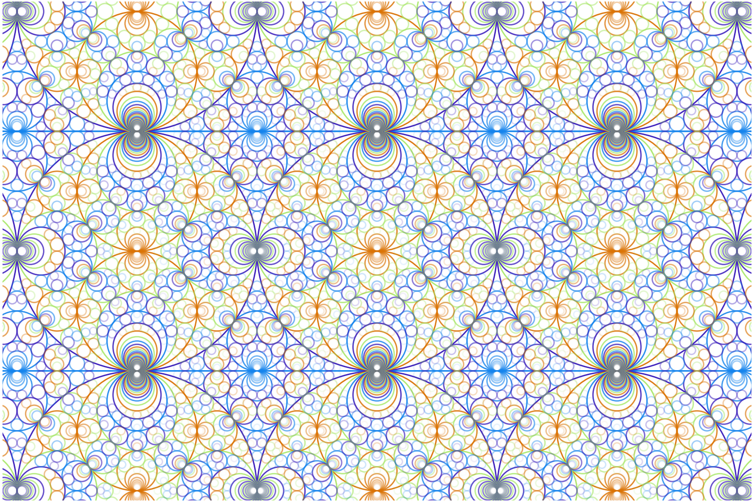
sized by norm of the denominator

3-dimensional Schmidt arrangement of $\mathbb{Q}(i)$



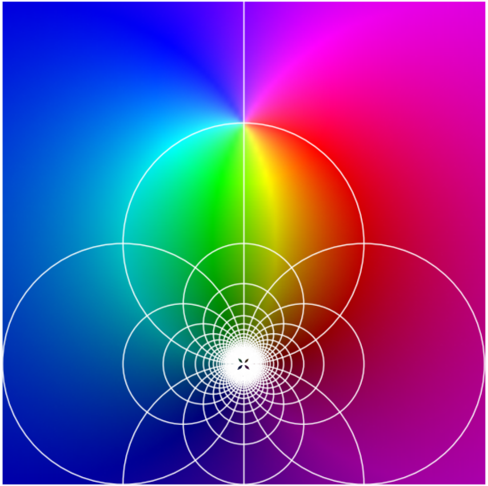


Schmidt arrangement of $\mathbb{Q}(i)$



orbit of real line under $\mathrm{PSL}_2(\mathbb{Z}[i])$

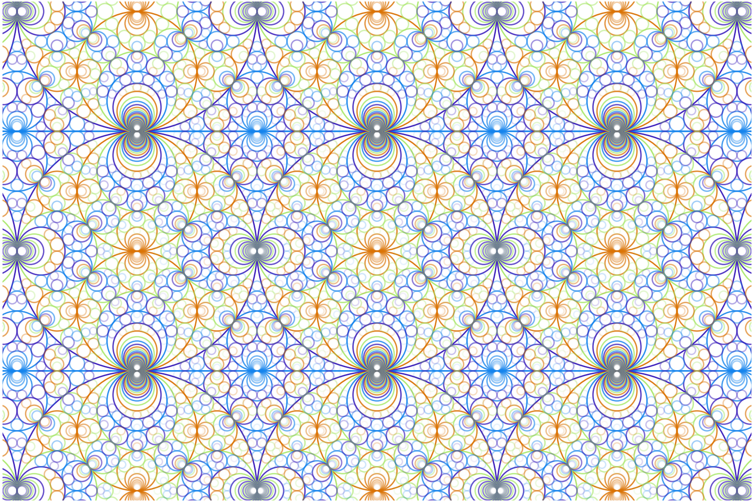
the language for circles: Möbius transformations



$\mathrm{PSL}_2(\mathbb{C})$ acts on the extended complex plane, taking circle to circles:

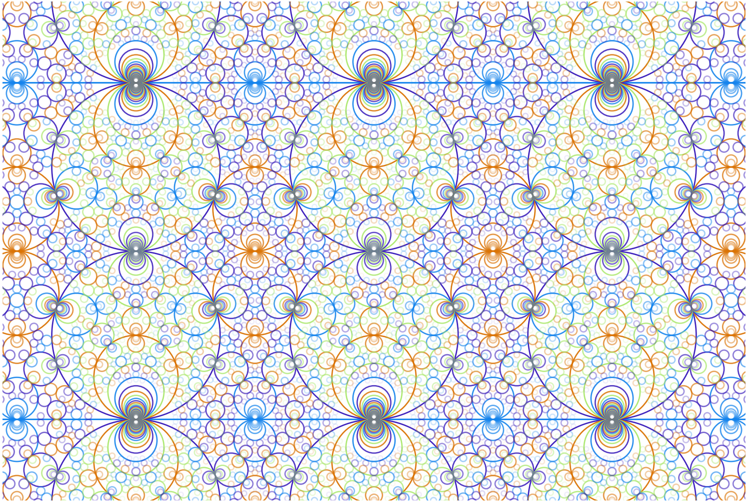
$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \quad \mapsto \quad \left(z \mapsto \frac{\alpha z + \gamma}{\beta z + \delta} \right)$$

Schmidt arrangement of $\mathbb{Q}(i)$



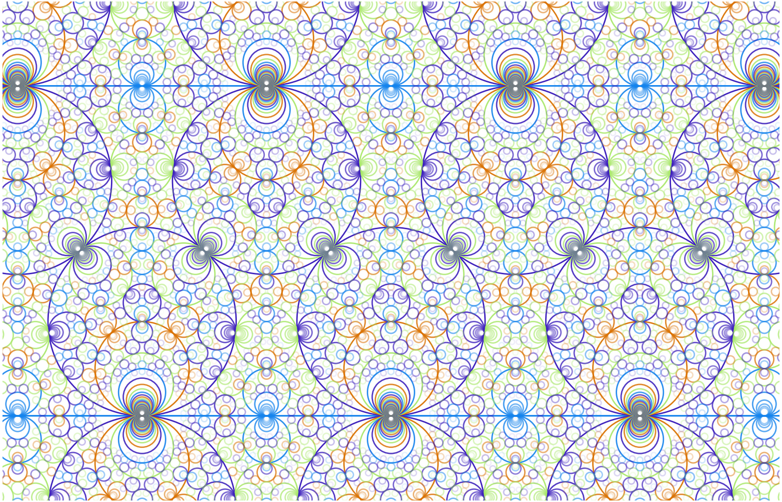
orbit of real line under $\mathrm{PSL}_2(\mathbb{Z}[i])$

Schmidt arrangement of $\mathbb{Q}(\sqrt{-2})$



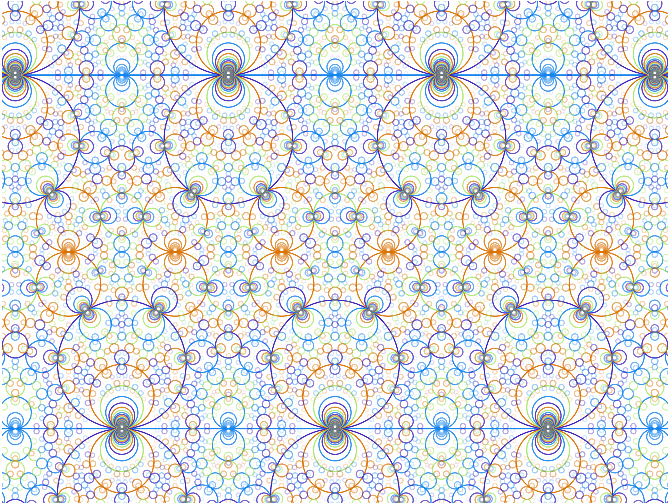
orbit of real line under $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-2}])$

Schmidt arrangement of $\mathbb{Q}(\sqrt{-7})$



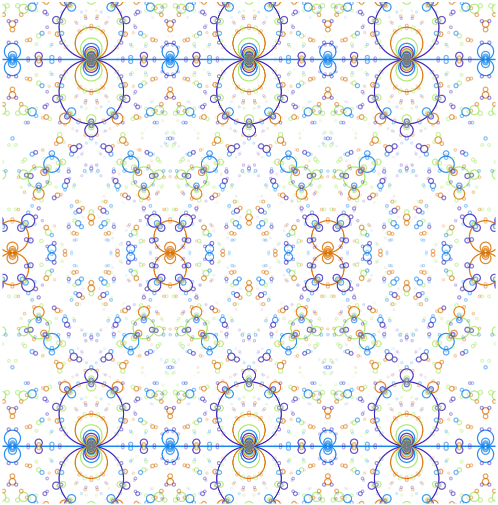
orbit of real line under $\mathrm{PSL}_2(\mathbb{Z}[\frac{1+\sqrt{-7}}{2}])$

Schmidt arrangement of $\mathbb{Q}(\sqrt{-11})$



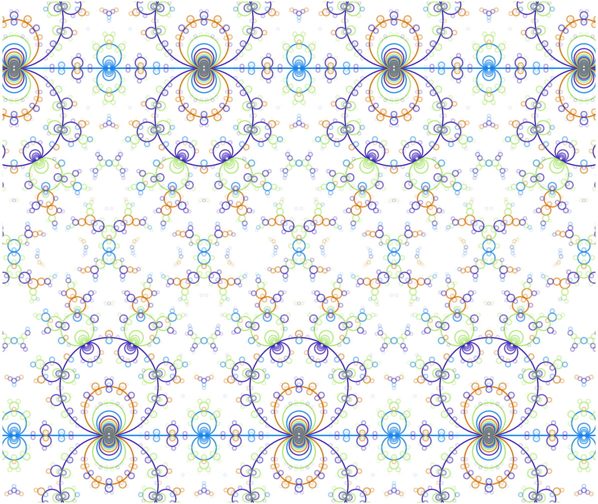
orbit of real line under $\mathrm{PSL}_2(\mathbb{Z}[\frac{1+\sqrt{-11}}{2}])$

Schmidt arrangement of $\mathbb{Q}(\sqrt{-6})$



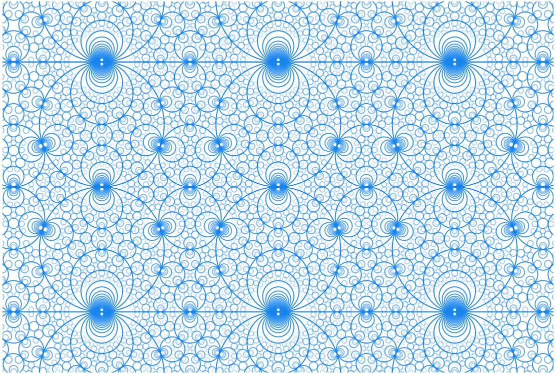
orbit of real line under $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-6}])$

Schmidt arrangement of $\mathbb{Q}(\sqrt{-15})$



orbit of real line under $\mathrm{PSL}_2(\mathbb{Z}[\frac{1+\sqrt{-15}}{2}])$

Schmidt arrangements

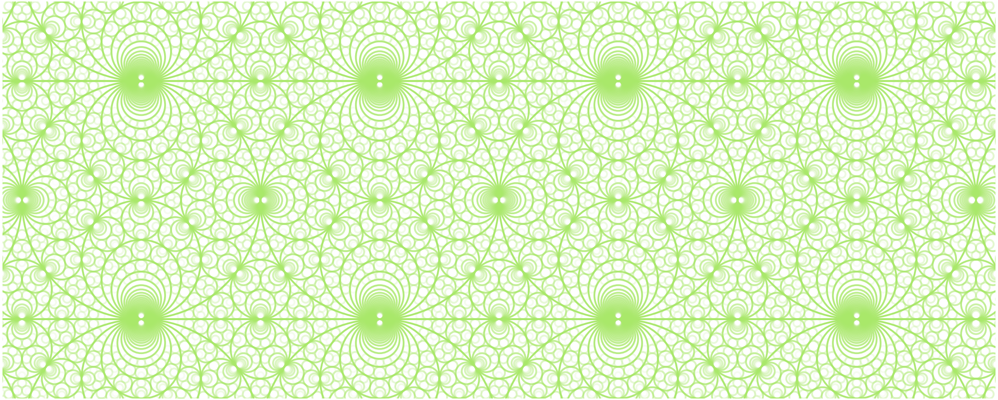


tangency points = rational points of trivial class

$\alpha/\beta \in K$ such that (α, β) is principal

size of the pencil $= 1/N(\beta)$

continued fractions: $\mathbb{Q}(i)$ in $\mathbb{C}$



Asmus Schmidt, 1975: continued fractions by zooming in through the regions
tangency points of the 'moves' are good approximations

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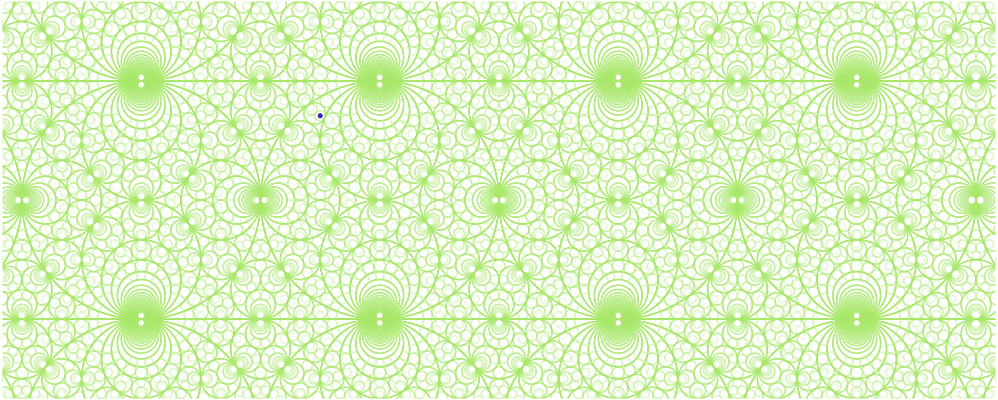

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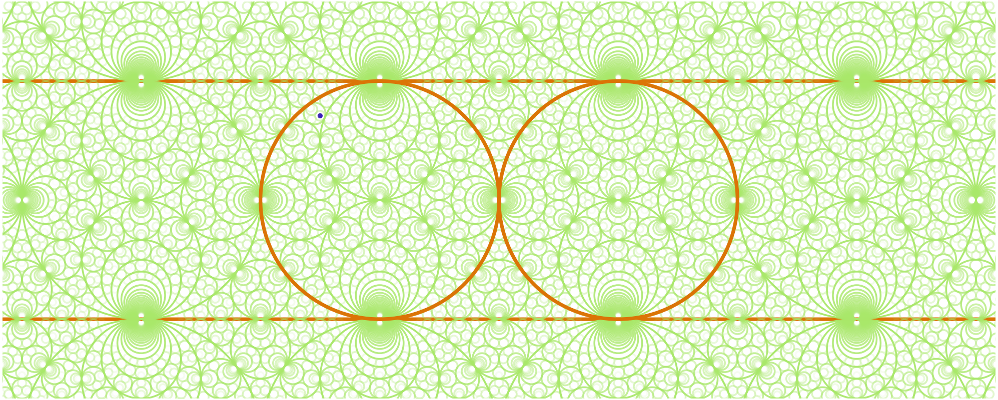

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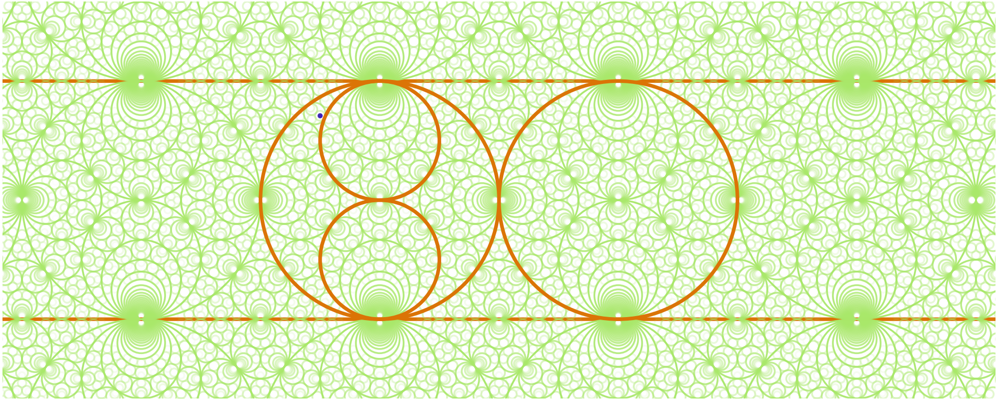

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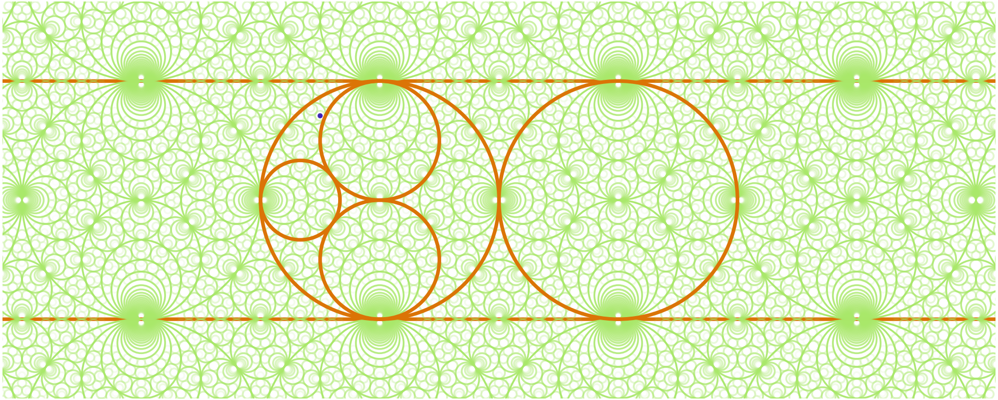

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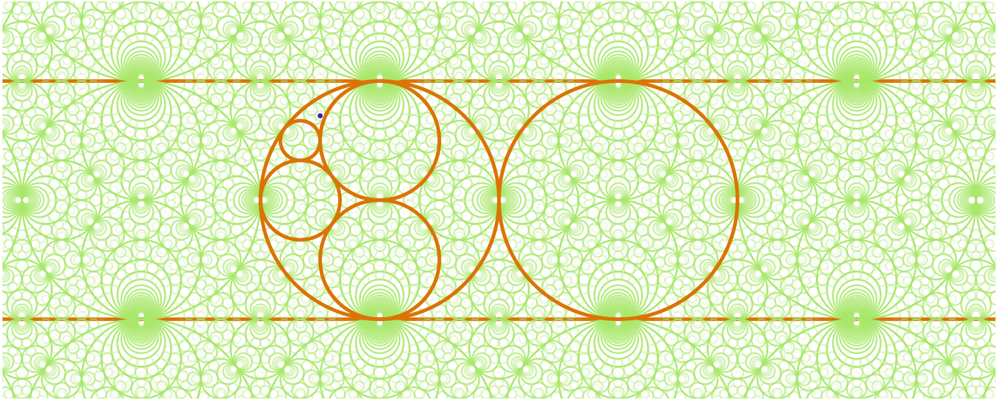

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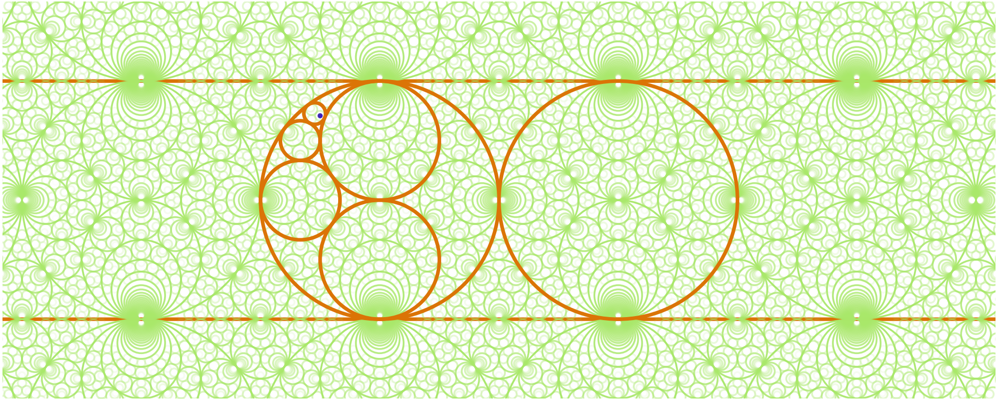

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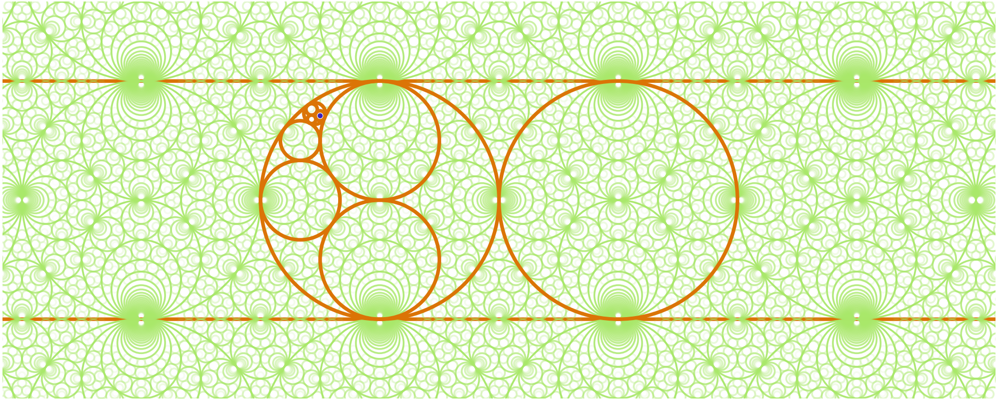

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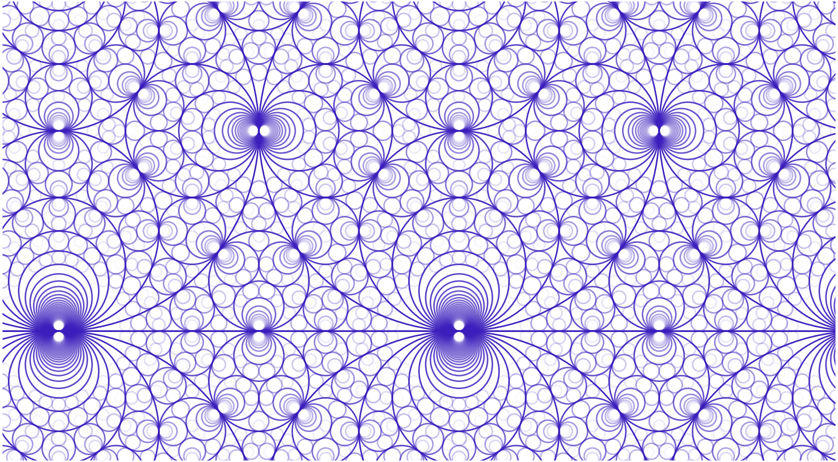
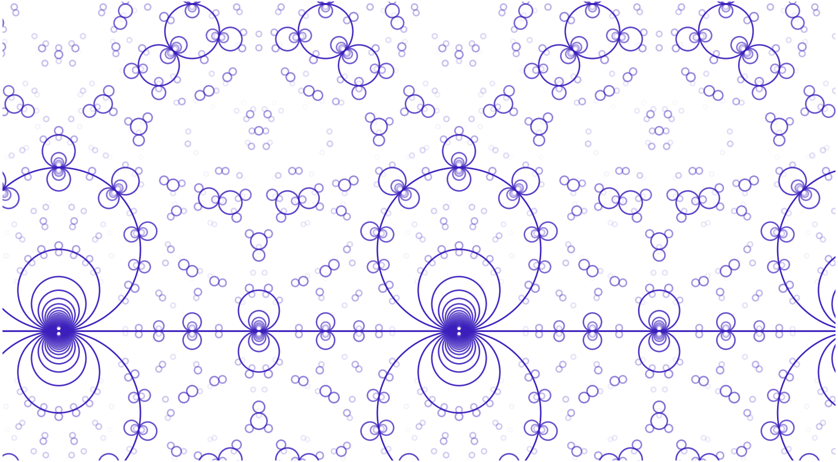
Euclideanity



Theorem (S.) Euclideanity = tangency (or topological) connectedness

Euclideanity



Theorem (S.) Euclideanity = tangency (or topological) connectedness

Euclideanity

The *tangency graph* G_K of a Schmidt arrangement is:

$$\left\{ \begin{array}{rcl} \textit{vertices} & = & \textit{circles} \\ \textit{edges} & = & \textit{tangencies} \end{array} \right\}.$$

Theorem (S.)

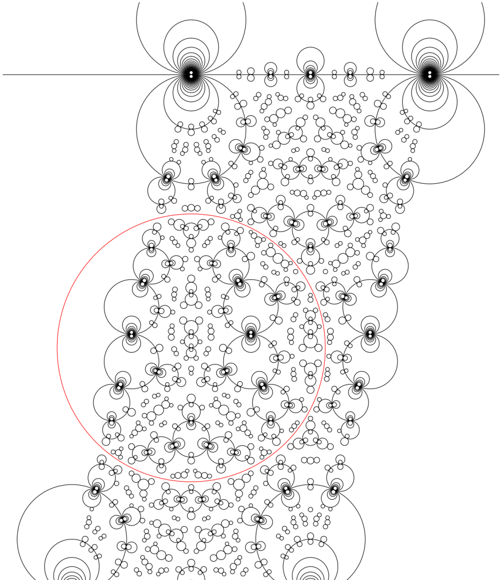
G_K is connected if and only if $\mathcal{O}_K$ is Euclidean.

Proof.

1. Connected component of $\widehat{\mathbb{R}}$ is all circles reachable by combinations of elementary matrices.
2. Thm of P.M. Cohn: $\mathcal{O}_K$ is Euclidean if and only if $\mathrm{SL}_2(\mathcal{O}_K)$ is generated by elementary matrices.

□

Euclideanity



Theorem (S.)

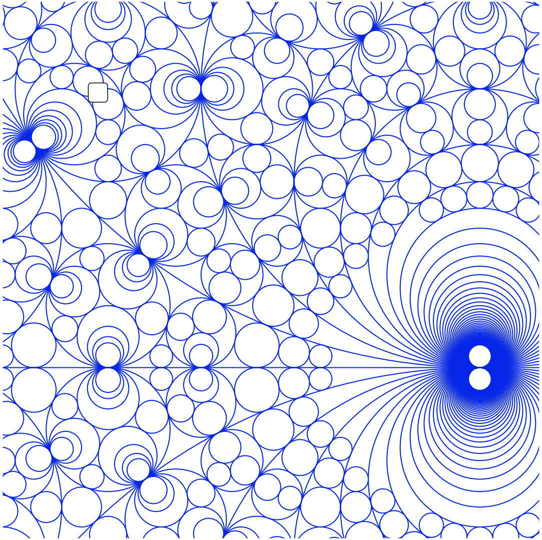
The Schmidt arrangement of K is connected if and only if $\mathcal{O}_K$ is Euclidean.

The *ghost circle* is the circle orthogonal to the unit circle having center

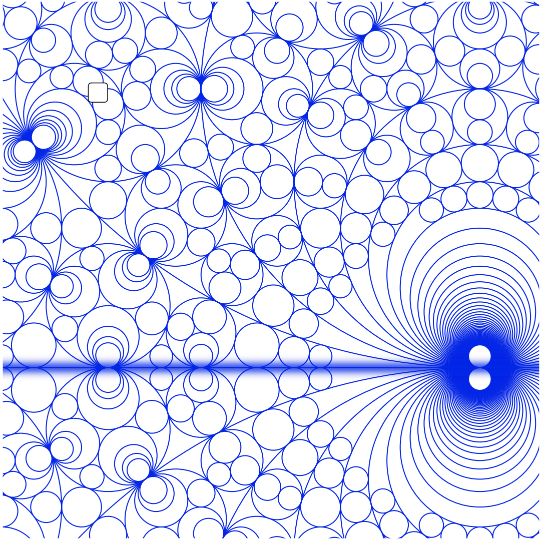
$$\begin{cases} \frac{1}{2} + \frac{\sqrt{\Delta}}{4} & \Delta \equiv 0 \pmod 4 \\ \frac{1}{2} + \frac{-\Delta - 1}{4\sqrt{\Delta}} & \Delta \equiv 1 \pmod 4 \end{cases}.$$

It exists only when $\mathcal{O}_K$ is non-Euclidean.

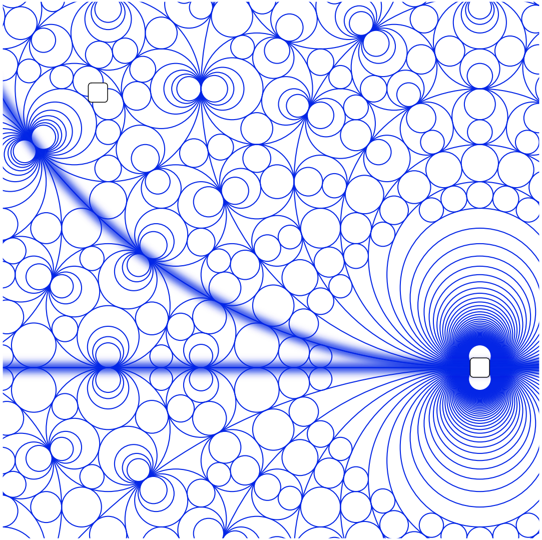
Continued fractions and connectivity



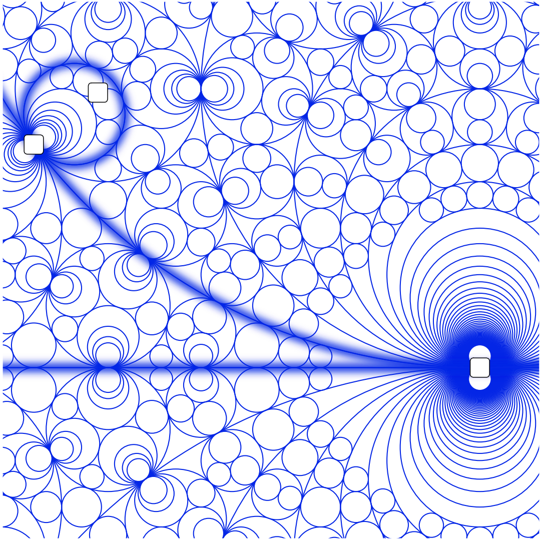
Continued fractions and connectivity



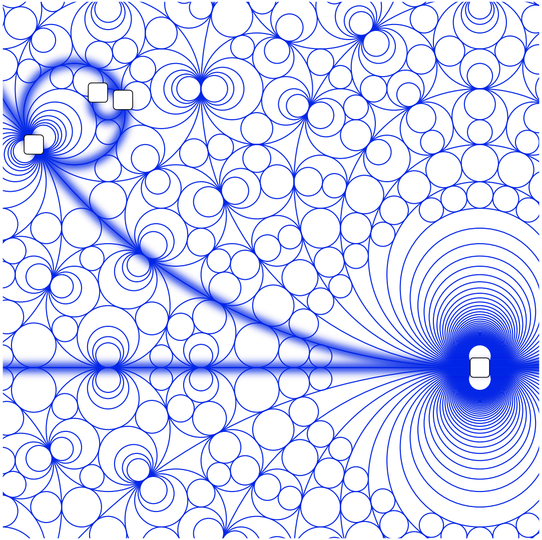
Continued fractions and connectivity



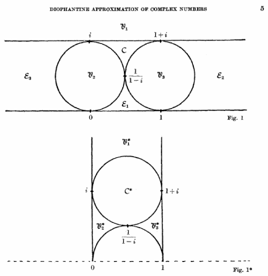
Continued fractions and connectivity



Continued fractions and connectivity



Continued fractions for the five Euclidean fields



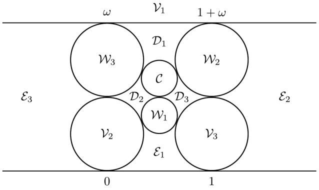


Fig. 1. ($\mathcal{I}_1$).

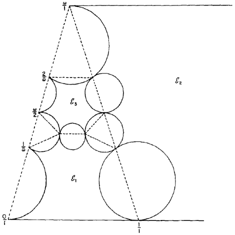
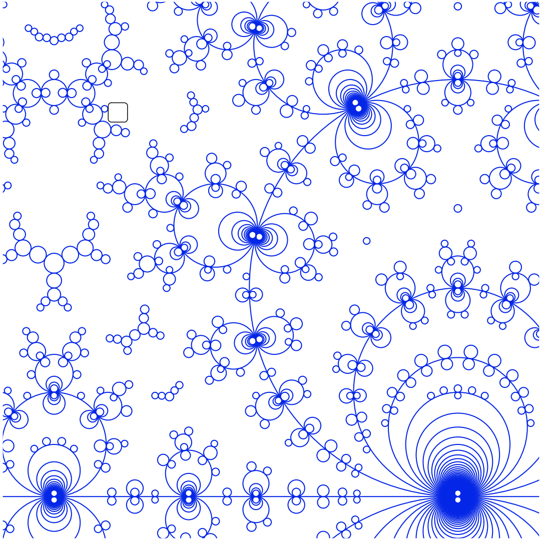


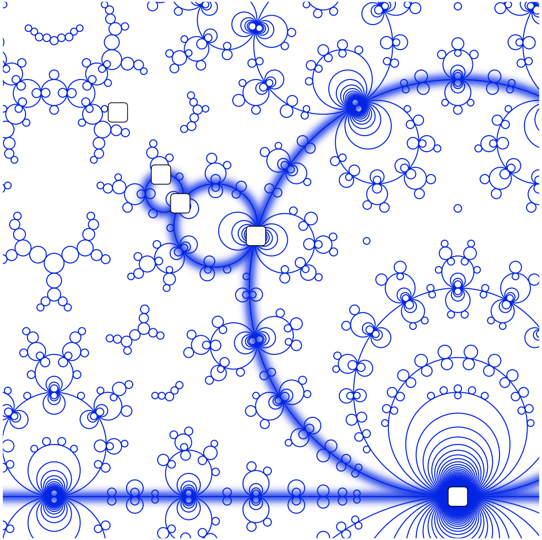
FIGURE 4

Asmus Schmidt (1975-2011): $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-7})$ and $\mathbb{Q}(\sqrt{-11})$.

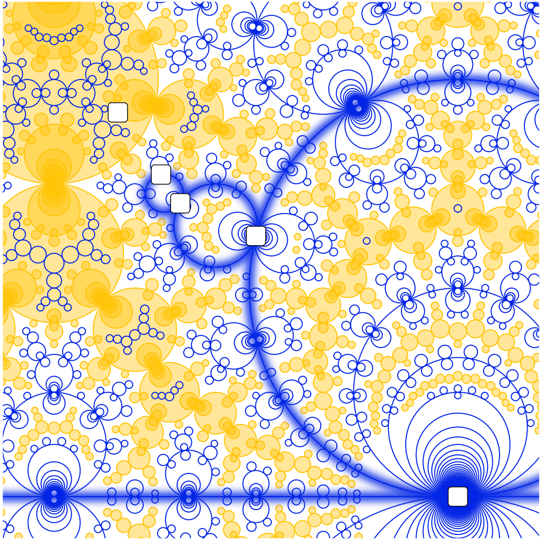
Continued fractions and connectivity



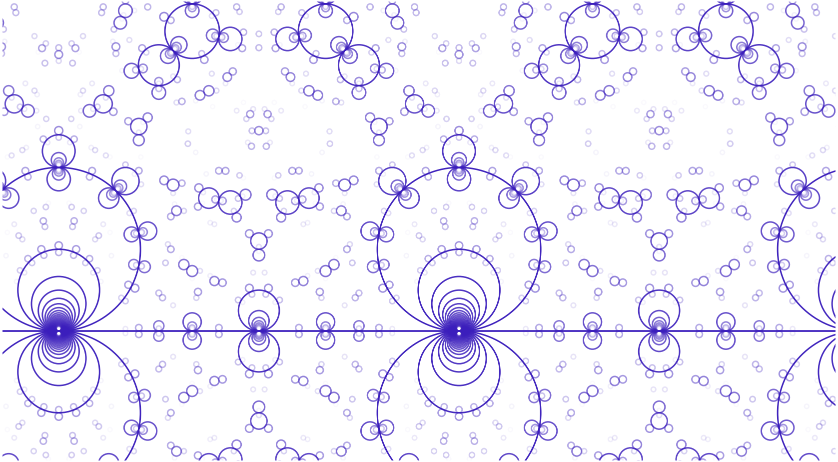
Continued fractions and connectivity



Continued fractions and connectivity



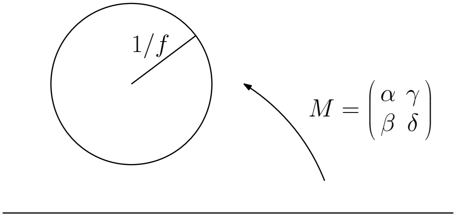
Question A: Non-Euclidean continued fractions?



Can we recover continued fractions when it is not connected?

The lattice associated to a circle

A circle is obtained by an element of $\mathrm{PSL}_2(\mathcal{O}_K)$.

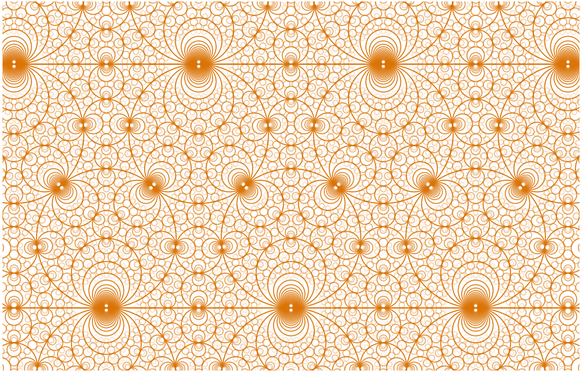


Consider the lattice $\Lambda = \beta\mathbb{Z} + \delta\mathbb{Z}$.
The *order* of Λ is

$$\{r \in \mathcal{O}_K : r\Lambda \subset \Lambda\}.$$

Then the order of Λ is an order in $\mathcal{O}_K$.
The orders of $\mathbb{Z}[\tau]$ are $\mathbb{Z}[f\tau]$, where $f \in \mathbb{N}$ is the *conductor*.
It so happens f is also the curvature of the circle!

Bijection: ideal classes of $\mathbb{Z}[f\tau]$ with circles of curvature f

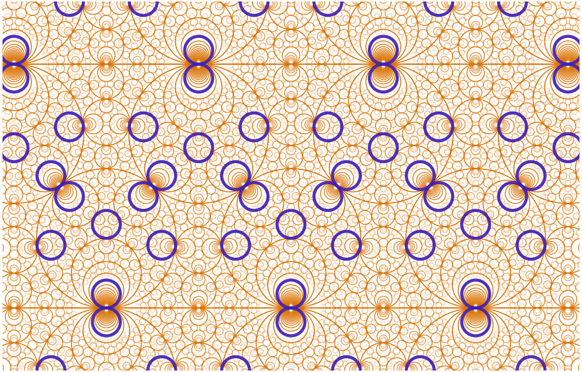


circles = ideal classes of orders
(which are trivial when extended to maximal ideal)

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \quad \longleftrightarrow \quad \beta\mathbb{Z} + \delta\mathbb{Z}$$

curvature of circle = conductor of the order

Bijection: ideal classes of $\mathbb{Z}[f\tau]$ with circles of curvature f



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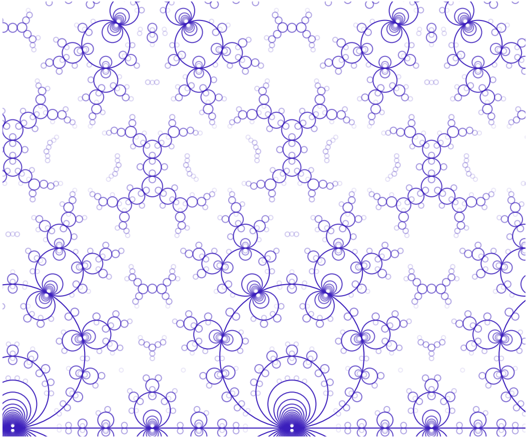
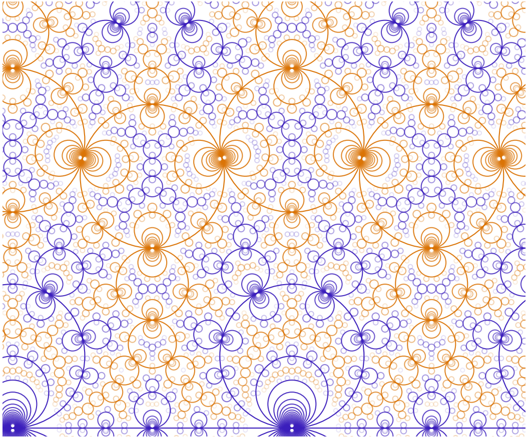


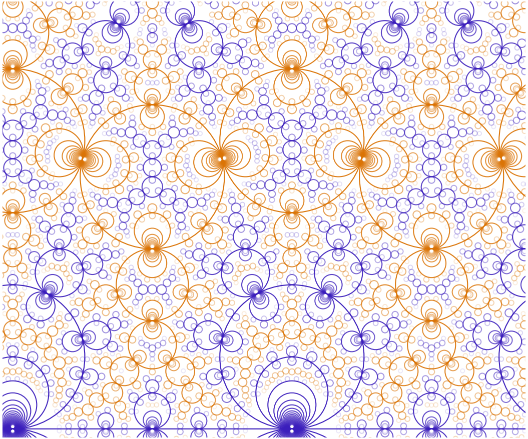
2-part of the class group = maximal discrete extension of $\mathrm{PSL}_2(\mathcal{O}_K)$

Bijection: ideal classes of $\mathbb{Z}[f\tau]$ with circles of curvature f



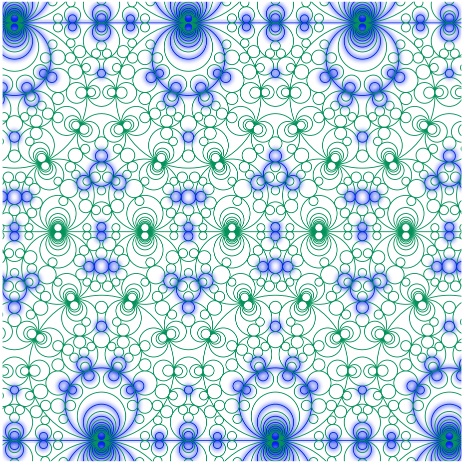
2-part of the class group = maximal discrete extension of $\mathrm{PSL}_2(\mathcal{O}_K)$

Question B: non-principle ideal classes?



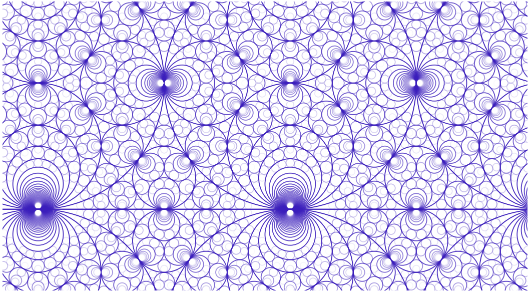
Is it possible to see the rest of the class group?

Extended Schmidt arrangements



Daniel Martin has found a way to see the rest of the class group and recover continued fractions.

Drawing Schmidt arrangements



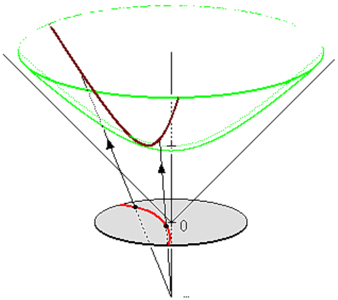
Theorem (Martin)

The circles of the Gaussian Schmidt arrangement are exactly those of center $(x/b, y/b)$ *and curvature* b *such that*

$$x^2 + y^2 \equiv 1 \pmod{4b}$$

Minkowski space and the hyperboloid model

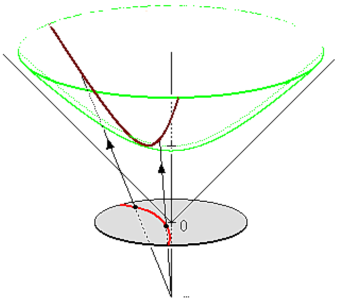
$$x^2 + y^2 + z^2 - t^2$$



vector outside the light cone	$\longleftrightarrow$	plane slicing the light cone	$\longleftrightarrow$	hyperbolic geodesic line (plane)
lattice points outside the light cone	$\longleftrightarrow$	planes slicing the light cone	$\longleftrightarrow$	circle packing

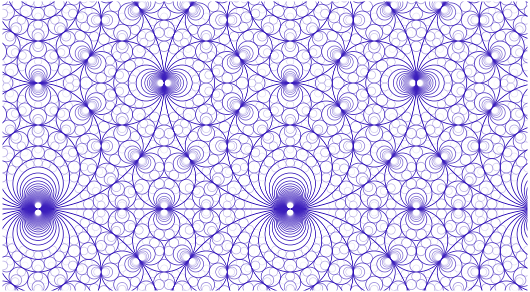
Minkowski space and the hyperboloid model

$$x^2 + y^2 + 2bb'$$



vector outside the light cone	$\longleftrightarrow$	plane slicing the light cone	$\longleftrightarrow$	hyperbolic geodesic line (plane)
lattice points outside the light cone	$\longleftrightarrow$	planes slicing the light cone	$\longleftrightarrow$	circle packing

Drawing Schmidt arrangements

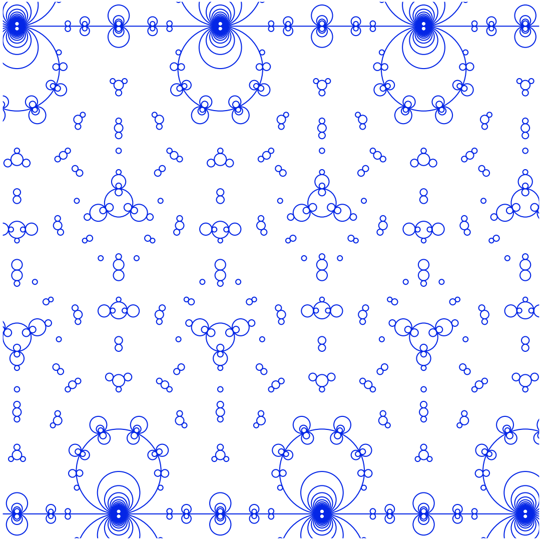


Theorem (Martin)

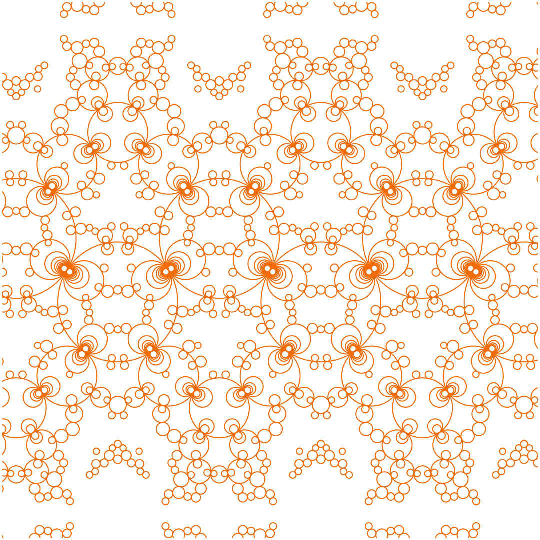
The circles of the Gaussian Schmidt arrangement are exactly those arising from vectors $\mathbf{v} := (x, y, b, b')$ *with* $|\mathbf{v}|^2 = 1$ *and* b *even.*

In other words, the intersection of a lattice with the one-sheeted hyperboloid $|\mathbf{v}|^2 = 1$.

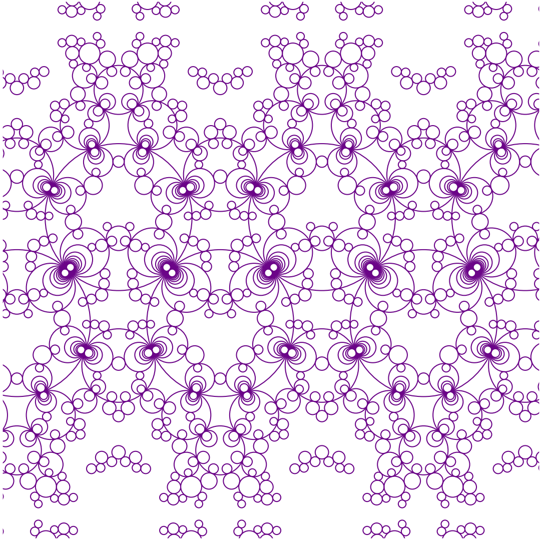
The sheet $|\mathbf{v}|^2 = 1$ for $\mathbb{Q}(\sqrt{-23})$



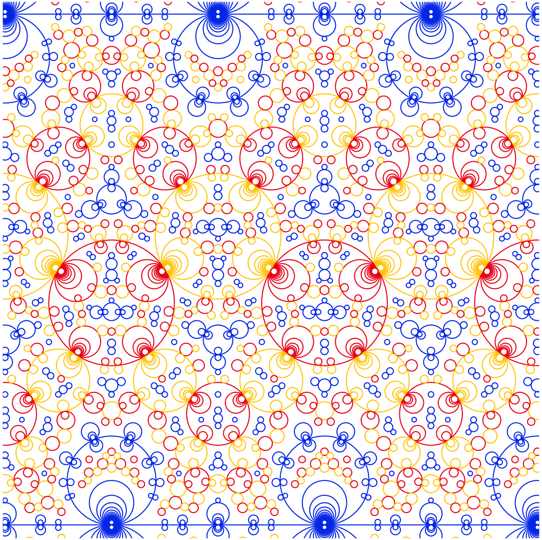
The sheet $|\mathbf{v}|^2 = 2$ for $\mathbb{Q}(\sqrt{-23})$



The sheet $|\mathbf{v}|^2 = 3$ for $\mathbb{Q}(\sqrt{-23})$



The sheet $|\mathbf{v}|^2 = 1$ and 2 for $\mathbb{Q}(\sqrt{-23})$



Extended Schmidt arrangements

It is possible to define (Martin) a set of matrices giving these as an orbit:

$$\mathscr{M}_D = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \left\| \frac{\det(M)}{(a,b,c,d)^2} \right\| = D \right\}$$

Extended continued fractions

Theorem (Martin)

There is a continued fraction algorithm, formed by stepping from circle to circle in these arrangements, so that the tangency points p_n/q_n along the way are good approximations in all the classical senses:

- $|z - p_n/q_n| < c/|a_n q_n^2|$
- $|q_n z - p_n| < \varepsilon |q_{n-1} z - p_{n-1}|$
- $|q_n| > 1/\varepsilon^n$
- *If* $p, q \in \mathcal{O}_K$ *with* $|q| < c''|q_n|$, *then* $|q_n z - p_n| < c'|qz - p|$.

The full class group in $\mathbb{Q}(\sqrt{-71})$