A bound for the number of automorphisms of an arithmetic Riemann surfaces

Exposition of a paper by Mikhail Belolipetsky and Gareth Jones

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Consider a Riemann surface as a quotient of $\mathcal{H}$ by its surface group.

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Then its automorphisms can be obtained from the automorphisms of $\mathcal{H}$:

$$\begin{aligned}\mathrm{Aut}(\mathcal{S}) &= \{\alpha \in \mathrm{PSL}(2, \mathbb{R}) : \alpha \Gamma_S \alpha^{-1} = \Gamma_S\}/\Gamma_S \\ &= N(\Gamma_S)/\Gamma_S\end{aligned}$$

(Think: Given $\gamma \in \Gamma_S$, we need $\alpha(\gamma(x)) = \gamma'(\alpha(x))$ for some $\gamma' \in \Gamma_S$.)

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Then, if we determine $\mathcal{S}$ by $\Gamma_S = K$, we have

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We call this a *surface-kernel epimorphism* or SKE.

To verify that the kernel is torsion free, we must check that every element of Γ of finite order has its order preserved by $p : \Gamma \to G$.

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Then we know that G is a subgroup of $\mathrm{Aut}(\mathcal{S})$.

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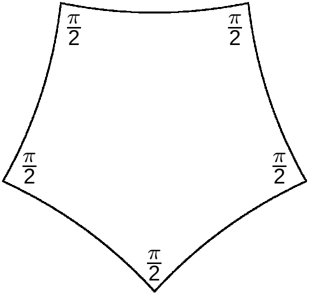
Arithmetic:

$$\begin{aligned} &(2,3,n), \quad n = 7,8,9,10,11,12,14,16,18,24,30 \\ &(2,4,n), \quad n = 5,6,7,8,9,10,12,18 \\ &(2,5,n), \quad n = 5,6,8,10,20,30 \\ &\qquad\qquad etc. \end{aligned}$$

K. Takeuchi. Arithmetic triangle groups. *J. Math. Soc. Japan* **29** (1977), 91-106.

Consider the right-angled hyperbolic pentagon:



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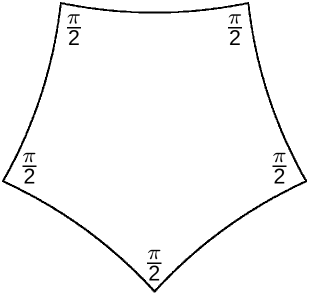
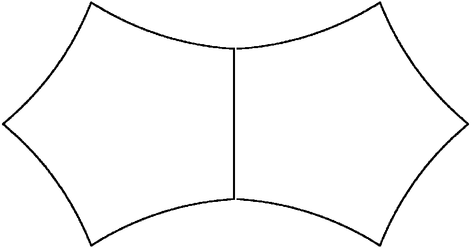


Let Γ be the orientation-preserving subgroup of the group of reflections in its sides.

The fundamental domain for Γ is two copies of the pentagon:



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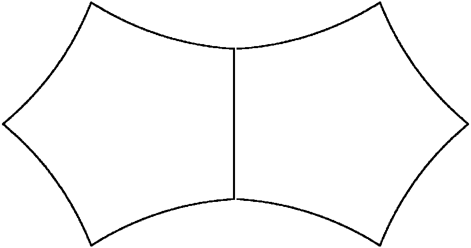

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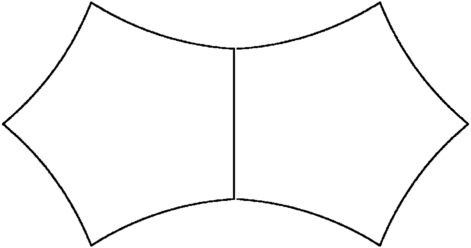

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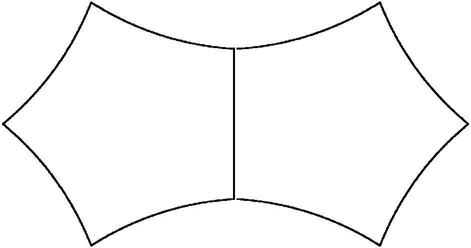

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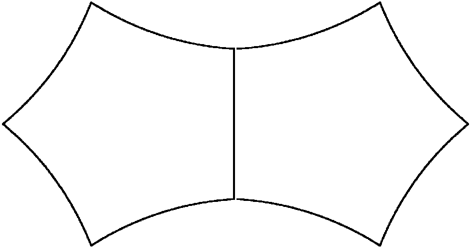

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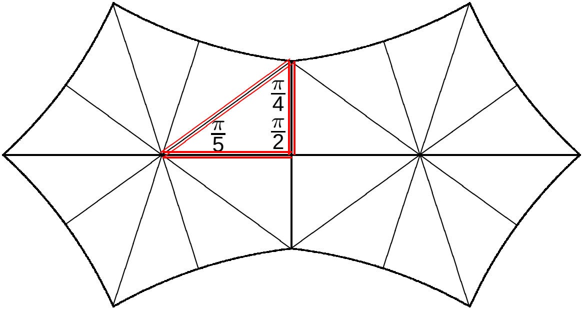
- Two reflections give rotation around an angle of π. This is order 2. There are five such elements of Γ.
- The signature of the group Γ is $(2, 2, 2, 2, 2)$.
- The Riemann surface $\mathcal{S} = \Gamma\backslash\mathcal{H}$ is of genus zero.

Subdivide the pentagon into 10 congruent triangles:



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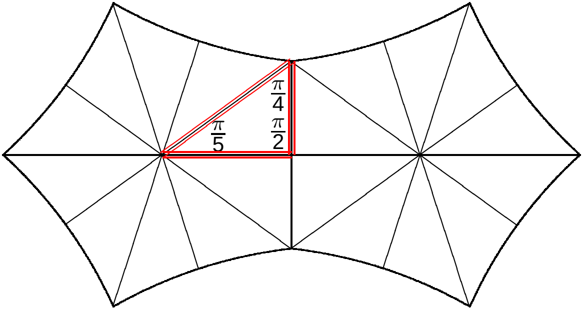

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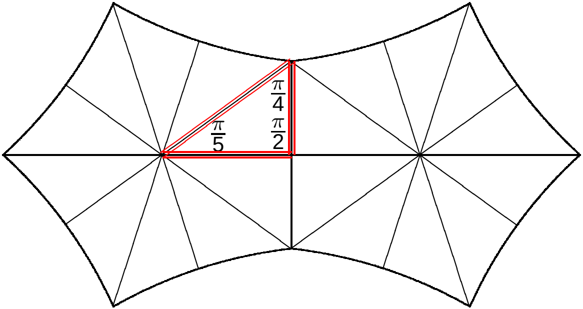

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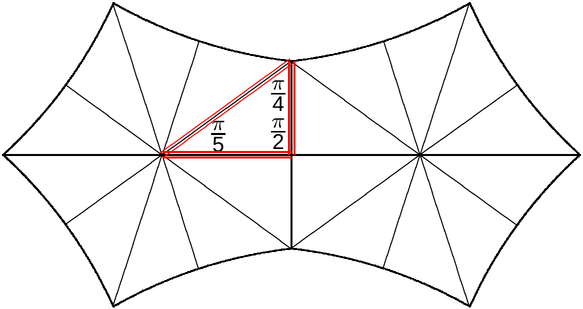

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- The triangle has angles $\pi/2$, $\pi/4$ and $\pi/5$. So Γ' is the $(2, 4, 5)$ triangle group, which is arithmetic.
- But Γ is a subgroup of Γ' of index 10. Hence the two groups are commensurable, and so Γ is arithmetic.

Outline

Lemma

Let $\{\mathcal{S}_g\}_{g\in\mathcal{G}}$ be an infinite sequence of arithmetic surfaces of different genera g, such that for each $g \in \mathcal{G}$, the group of automorphisms of $\mathcal{S}_g$ has order $a(g+b)$ for some fixed a and b. Then $b = -1$.

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Proof. Let $\mathcal{S}$ be a surface from the given sequence.

Then $\mathrm{Aut}(\mathcal{S}) \cong N(\Gamma_{\mathcal{S}})/\Gamma_{\mathcal{S}}$, where $\Gamma_{\mathcal{S}}$ is the surface group corresponding to $\mathcal{S}$.

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The Riemann-Hurwitz formula yields

$$\mu(N(\Gamma_\mathcal{S})) = \frac{\mu(\Gamma_\mathcal{S})}{|\mathrm{Aut}(\mathcal{S})|} = \frac{2\pi(2g-2)}{a(g+b)} ,$$

so $\mu(N(\Gamma_\mathcal{S})) \to 4\pi/a$ as $g \to \infty$.

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So for all but finitely many $g \in \mathcal{G}$,

$$\frac{2\pi(2g-2)}{a(g+b)} = \mu(N(\Gamma_{\mathcal{S}})) = \frac{4\pi}{a} \, .$$

Therefore $b = -1$.

It follows from Lemma 1 that the Accola-Maclachlan lower bound for $N(g)$, $8(g+1)$, cannot be attained by infinitely many arithmetic surfaces.

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In fact it is never attained by arithmetic surfaces, since the extremal surfaces for this bound are uniformized by surface subgroups of (2, 4, 2(g+1))-groups with $g \geq 24$ (Maclachlan), and these are not arithmetic (Takeuchi).

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θ is a SKE and thus $K = \ker(\theta)$ is a surface group.

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Then $N_{ar}(g) \geq |\mathrm{Aut}(\mathcal{S})| \geq |G| = 4(g-1)$ as required.

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- Imposing specific conditions on g we a get a contradiction.
- Show that infinitely many values of g satisfy these conditions. For these $N_{\text{ar}}(g) = 4(g-1)$.

By our hypothesis, $G \cong \Gamma/K$ for some co-compact arithmetic group Γ and normal surface subgroup $K = \Gamma_{\mathcal{S}}$ of Γ, with

$$4\pi(g-1) = \mu(K) = |G|\mu(\Gamma) > 4(g-1)\mu(\Gamma), \qquad (1)$$

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For each $\sigma \in \Sigma$, the number $q = \frac{\mu(\Gamma)}{4\pi}$ is rational and depends only on the signature σ of Σ, so writing $q = r/s = r_\sigma/s_\sigma$ in reduced form, we have $|G| = (g-1)/q = (g-1)s/r$.

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Since $|Q|$ is coprime to p, the natural epimorphism $G \to Q$ preserves the orders of the images of all elliptic generators of Γ.

The inclusions $K \trianglelefteq \Delta \trianglelefteq \Gamma$ induce an étale $\mathbb{Z}/p\mathbb{Z}$-covering of Riemann surfaces

$$\begin{tikzcd}
\mathcal{S} \cong K \setminus \mathcal{H} \arrow[d, "P \cong \mathbb{Z}/p\mathbb{Z}"'] \arrow[dd, bend left=60, dash, "G"] \\
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Then Q is a group of automorphisms of a Riemann surface $\mathcal{T}$ of genus 2.

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In particular $E \equiv 0 \pmod 2$.

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- We have infinitely many p satisfying our conditions.

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Therefore we have a map $Q \to \mathrm{GL}_1(\mathbb{F}_p)^4$.

Lemma

Lemma (Farkas & Kra, V.3.4) If $A \in \mathrm{SL}_k(\mathbb{Z})$ *has finite order* $m > 1$ *and* $A \equiv I \pmod{n}$ *then* $m = n = 2$.

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ϵ thus divides $\gcd(E, p - 1)$.

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This contradicts $0 < \mu(\Gamma) < \pi$.

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Therefore we have an infintely many g that lead to a contradiction.

Outline

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  Assume that $g - 1 =: p$ is a prime such that $\gcd(p, R) = 1$, $p \notin S$, $p > S$ and such that $\gcd(p - 1, E) = 2$, where E is the least common multiple of the exponents of all automorphism groups of Riemann surfaces of genus 2.

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  Then the size of the automorphism group of any surface of genus g cannot be greater than $4(g - 1)$, so we have to have equality.

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Construct a specific sequence of genera g such that N_{ar} attains the lower bound.

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Theorem (Main Theorem)

For all primes $p \equiv 23, 47, 59 \pmod{60}$, we have $N_{ar}(g) = 4(g-1)$. The least genus g for which the the lower bound $N_{ar}(g) = 4(g-1)$ is attained is $g = 24$.

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Idea

Construct primes p satisfying the hypotheses of the Main Theorem. Then $g = p + 1$ will be such that:

$$N_{ar}(g) = 4(g-1).$$

Strategy

1. Listing all Arithmetic Fuchsian Signatures
2. The Conditions on Sufficiently Large Primes p
3. Smaller Primes

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- Then by the proof of the Main Theorem, for any prime p not dividing R, not contained in Π and greater than S, we cannot have

$$|G| > 4(g-1)$$

if we impose the additional condition that $\gcd(p-1, E) = 2$.

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- Let $(g; m_1; \ldots; m_r)$ be the signature of a Fuchsian group Γ. Then

$$\frac{1}{\pi}\mu(\Gamma) = 4(g-1) + \sum_{k=1}^{r}\left(1 - \frac{1}{m_k}\right) < 1 \tag{2}$$

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- If $g = 0$, then since $m_k \geq 2$, we must have $r < 5$, so all signatures have length 3 or 4.
- Takeuchi gave a complete list of cocompact arithmetic triangle groups; almost all of these have volume less than π.

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- The only other possible candidates are $(2,2,3,3)$,$(2,2,3,4)$,$(2,2,3,5)$ and $(2,2,2,n)$, for $n \geq 3$.

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- It can be shown that there are only 12 signatures for which $(2,2,2,n)$ is arithmetic.

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- Further examining the list of possible signatures, and putting $\frac{\mu(\Gamma)}{4\pi}$ into lowest terms, we find that $R = 4 \cdot 3 \cdot 5 \cdot 7$ is the least common multiple of the numerators of all $\frac{\mu(\Gamma_\sigma)}{4\pi}$ and $s = 84$ is the largest occurring denominator.

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Lemma

If S is a Riemann surface of genus $\gamma \geq 2$, then it has no automorphisms of prime order greater than $2\gamma + 1$.

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- for $\gamma' = 1$, $2(\gamma - 1) = m(p - 1) \geq p - 1 \geq 2\gamma - 1$, a contradiction.

$\square$

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- For $\gamma' = 0$, $2(\gamma - 1) = -2pg + m(p-1)$, we have $m = \frac{2\gamma}{p-1} + 2 \leq \frac{p}{p-1} + 2 \leq 3$, so $m = 3$.

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- **Conclusion:** No prime other than $\{2, 3, 5\}$ divides E, the least common multiple of the exponents of automorphism groups of surfaces of genus 2. Thus the condition that $\gcd(p - 1, E) = 2$ is satisfied by all p such that $p - 1$ is not divisible by $3, 4, 5$.

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- Since we also require that $p \not\equiv 0 \mod q$ for $q = 2,3,5$, this leaves the possibilities that $p \equiv 2 \pmod 3$, $p \equiv 3 \mod 4$ and $p \equiv 2, 3, 4 \mod 5$. The first two lift to the congruence $p \equiv 11 \pmod{12}$; combining with the last one gives $p \equiv 23, 47, 59 \pmod{60}$ as the equivalent congruence.

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- Thus, surfaces of genus $p + 1$ for any such p satisfy the lower bound: $N_g = 4(g - 1)$.
- What about $p = 23, 47, 59$ or 83?

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- 59 is coprime to R, so $|Aut(S)| = |G| = (g-1)s = 59s$ for some s.
- By inspection, s is coprime to 59, so a 59-Sylow subgroup is of order 59. Letting n_{59} be the number of 59-Sylow subgroups, we must have $n_{59}|s$ and $n_{59} \equiv 1 \pmod{59} \Rightarrow n_{59} = 1$. So the 59-Sylow subgroup P_{59} is unique.

Smaller Primes: p=59

- $p = 59$, S of genus $g = 60$:
- 59 is coprime to R, so $|Aut(S)| = |G| = (g-1)s = 59s$ for some s.
- By inspection, s is coprime to 59, so a 59-Sylow subgroup is of order 59. Letting n_{59} be the number of 59-Sylow subgroups, we must have $n_{59}|s$ and $n_{59} \equiv 1 \pmod{59} \Rightarrow n_{59} = 1$. So the 59-Sylow subgroup P_{59} is unique.
- $p \notin \Pi = \{2, 3, 5, 7\}$, the set of primes dividing an element of order in some Γ_σ.

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- $p = 59$, S of genus $g = 60$:
- 59 is coprime to R, so $|Aut(S)| = |G| = (g-1)s = 59s$ for some s.
- By inspection, s is coprime to 59, so a 59-Sylow subgroup is of order 59. Letting n_{59} be the number of 59-Sylow subgroups, we must have $n_{59}|s$ and $n_{59} \equiv 1 \pmod{59} \Rightarrow n_{59} = 1$. So the 59-Sylow subgroup P_{59} is unique.
- $p \notin \Pi = \{2, 3, 5, 7\}$, the set of primes dividing an element of order in some Γ_σ.
- $p - 1 = 58 = 2 \cdot 19$, so $\gcd(p-1, E) = 2$.

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- 59 is coprime to R, so $|Aut(S)| = |G| = (g-1)s = 59s$ for some s.
- By inspection, s is coprime to 59, so a 59-Sylow subgroup is of order 59. Letting n_{59} be the number of 59-Sylow subgroups, we must have $n_{59}|s$ and $n_{59} \equiv 1 \pmod{59} \Rightarrow n_{59} = 1$. So the 59-Sylow subgroup P_{59} is unique.
- $p \notin \Pi = \{2, 3, 5, 7\}$, the set of primes dividing an element of order in some Γ_σ.
- $p - 1 = 58 = 2 \cdot 19$, so $\gcd(p-1, E) = 2$.
- Conclusion: $g = 60$ attains the lower bound.

Smaller Primes: p=83

- $p = 83$, S of genus $g = 84$:

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- $p = 83$, S of genus $g = 84$:
- 83 is coprime to R, so $|Aut(S)| = |G| = (g-1)s = 83s$ for some s. By inspection, s is coprime to 83, so if P_{83} is a 83-Sylow subgroup, then $|P_{83}| = 83$. Letting n_{83} be the number of 83-Sylow subgroups, we must have $n_{83}|s$ and $n_{59} \equiv 1 \pmod{59}$.

Smaller Primes: p=83

- $p = 83$, S of genus $g = 84$:
- 83 is coprime to R, so $|Aut(S)| = |G| = (g-1)s = 83s$ for some s. By inspection, s is coprime to 83, so if P_{83} is a 83-Sylow subgroup, then $|P_{83}| = 83$. Letting n_{83} be the number of 83-Sylow subgroups, we must have $n_{83}|s$ and $n_{59} \equiv 1 \pmod{59}$.
- Claim: P_{83} is normal in G.

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- $p = 83$, S of genus $g = 84$:
- 83 is coprime to R, so $|Aut(S)| = |G| = (g-1)s = 83s$ for some s. By inspection, s is coprime to 83, so if P_{83} is a 83-Sylow subgroup, then $|P_{83}| = 83$. Letting n_{83} be the number of 83-Sylow subgroups, we must have $n_{83}|s$ and $n_{59} \equiv 1 \pmod{59}$.
- Claim: P_{83} is normal in G.

Proof:

Smaller Primes: p=83

- $p = 83$, S of genus $g = 84$:
- 83 is coprime to R, so $|Aut(S)| = |G| = (g-1)s = 83s$ for some s. By inspection, s is coprime to 83, so if P_{83} is a 83-Sylow subgroup, then $|P_{83}| = 83$. Letting n_{83} be the number of 83-Sylow subgroups, we must have $n_{83}|s$ and $n_{59} \equiv 1 \pmod{59}$.
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Proof:

- The only possibility for the 83-Sylow subgroup P_{83} not being unique is if $n_{83} = s = 84$.

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- 83 is coprime to R, so $|Aut(S)| = |G| = (g-1)s = 83s$ for some s. By inspection, s is coprime to 83, so if P_{83} is a 83-Sylow subgroup, then $|P_{83}| = 83$. Letting n_{83} be the number of 83-Sylow subgroups, we must have $n_{83}|s$ and $n_{59} \equiv 1 \pmod{59}$.
- Claim: P_{83} is normal in G.

Proof:

- The only possibility for the 83-Sylow subgroup P_{83} not being unique is if $n_{83} = s = 84$.
- Then the normaliser of P_{83} is just P, so G acts faithfully and transitively on P_{83} (Frobenius action).
  $\Rightarrow$ There exists a normal subgroup N of G such that G is the semidirect product of N and P_{83}.

Smaller Primes: p=83,47,23

- In particular, there exists an epimorphism $G \to \mathbb{Z}_{83}$.

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- Also, $p - 1 = 82 = 2 \cdot 41$, so $\gcd(p - 1, E) = 2$.

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- Also, $p - 1 = 82 = 2 \cdot 41$, so $\gcd(p-1, E) = 2$. Therefore, $p = 83$ satisfies all required conditions to exclude that $|G| > 4(g-1) = 4 \cdot 83$.

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- Conclusion: $g = 60$ attains the lower bound.

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- Conclusion: $g = 60$ attains the lower bound.
- Similarly, one can show that for $p = g - 1 = 47$, there exists a unique normal subgroup of order 47, and satisfies the other conditions of the Main Theorem as well.

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- Conclusion: $g = 60$ attains the lower bound.
- Similarly, one can show that for $p = g - 1 = 47$, there exists a unique normal subgroup of order 47, and satisfies the other conditions of the Main Theorem as well.
- Using more results from group theory, one can show that $p = 23$ attains the lower bound as well.

Smaller Primes: p=83,47,23

- In particular, there exists an epimorphism $G \to \mathbb{Z}_{83}$.
- But since $s = 84$, $\Gamma = \Gamma(2,3,7)$ is a triangle group, this is impossible. Thus P_{83} must be normal as required.
- Also, $p - 1 = 82 = 2 \cdot 41$, so $\gcd(p-1, E) = 2$. Therefore, $p = 83$ satisfies all required conditions to exclude that $|G| > 4(g-1) = 4 \cdot 83$.
- Conclusion: $g = 60$ attains the lower bound.
- Similarly, one can show that for $p = g - 1 = 47$, there exists a unique normal subgroup of order 47, and satisfies the other conditions of the Main Theorem as well.
- Using more results from group theory, one can show that $p = 23$ attains the lower bound as well.
- In fact, one can show that $g = 24$ is the smallest prime such that $N_{ar}(g) = 4(g-1)$.

Explicit Sequence Theorem

Theorem (Explicit Sequence Theorem)

For all primes $p \equiv 23, 47, 59 \pmod{6}0$, *we have* $N_{ar}(g) = 4(g-1)$. *The least genus* g *for which the the lower bound* $N_{ar}(g) = 4(g-1)$ *is attained is* $g = 24$.

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