Stirling Numbers of the First Kind

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1 Getting to know the Stirling numbers

Definition 1. *The* (n, k) *signless Stirling number of the first kind, denoted* $c(n, k)$*, is equal to the number of permutations of* $[n]$ *having exactly* k *cycles.*

1. Compute the following:

   $c(3, 0) =$
   $c(3, 1) =$
   $c(3, 2) =$
   $c(3, 3) =$
   $c(3, 4) =$
   $c(4, 2) =$

2. Describe the general patterns:

   $c(n, 0) =$
   $c(n, 1) =$
   $c(n, n) =$
   If $k > n$, then $c(n, k) =$

2 Recurrence

Give a combinatorial proof of the following recurrence:

$$c(n, k) = c(n-1, k-1) + (n-1)c(n-1, k).$$

3 Generating Function

Let

$$G_n(x) := \sum_{k=0}^{\infty} c(n,k)x^k.$$

1. Why is $G_n(x)$ a polynomial?
2. Compute the following polynomials, and also give them in factored form

   $G_1(x) =$

   $G_2(x) =$

   $G_3(x) =$

3. Write a conjecture, based on the data above: *The generating function $G_n(x)$ is equal to the polynomial $P_n(x)$ defined by $P_n(x) =$*
4. Based on your definition of $P_n(x)$ above, give a recurrence relation for $P_n(x)$ in terms of $P_{n-1}(x)$.
5. Using the recurrence relation of the last section, conjecture a recurrence for $G_n(x)$ in terms of $G_{n-1}(x)$.

6. Verify this recurrence by direct computation using the definition of $G_n(x)$.

7. Use the work you've done to show that $G_n(x) = P_n(x)$.

4 Something else

Permutations are functions, so they can be composed.

1. Compose a few permutations to get the feel for it.

2. Does composition commute?

3. Let σ^k denote composition of a permutation σ with itself k times. Prove that for any permutation of $[n]$, there is some integer $k > 0$, so that σ^k is the identity.