

No more right-side left-side!
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Consider the following theorem:

[Theorem] For any real numbers a, b, q and r , the following relationship holds:

$$\frac{a-b}{a} = \left(\frac{q}{r}\right) \left(\frac{r}{q}\right) - \frac{b}{a}$$

Let's look at some left-side right-side proofs. What defines left-side right-side isn't that there are two sides, but that *you start with what you wish to prove and deduce something you know*. The first one is standard.

Right-side left-side 'proof' #1. We begin with what we want to show:

$$\begin{array}{ll} \frac{a-b}{a} = \left(\frac{q}{r}\right) \left(\frac{r}{q}\right) - \frac{b}{a} & \text{what we want} \\ \frac{a-b}{a} + \frac{b}{a} = \left(\frac{q}{r}\right) \left(\frac{r}{q}\right) & \text{add something to both sides} \\ \frac{a-b+b}{a} = \frac{qr}{rq} & \text{simplify} \\ \frac{a}{a} = \frac{qr}{qr} & \text{simplify} \\ 1 = 1 & \text{we're done!} \end{array}$$

□

The second one seems ok too, but there's a problem with this one. It's hard to say exactly what the problem is, though, since each step *seems* ok.

Right-side left-side 'proof' #2. We start with what we want to show:

$$\begin{array}{ll} \frac{a-b}{a} = \left(\frac{q}{r}\right) \left(\frac{r}{q}\right) - \frac{b}{a} & \text{what we want} \\ 0 \times \frac{a-b}{a} = 0 \times \left(\left(\frac{q}{r}\right) \left(\frac{r}{q}\right) - \frac{b}{a} \right) & \text{multiply something on both sides} \\ 0 = 0 & \text{we're done!} \end{array}$$

□

Can you see what's wrong? This method of proof could be used to prove something false! For example

A no-good nonsense non-proof that $2 = 3$.

$$\begin{array}{ll} 2 = 3 & \text{what we want} \\ 0 \times 2 = 0 \times 3 & \text{multiply something on both sides} \\ 0 = 0 & \text{we're done!} \end{array}$$

□

That last one illustrates the danger of the left-side right-side philosophy. The problem in this case is that each step wasn't 'reversible'. But in the direction we went (down the page), everything seemed legitimate. **Starting with what you want to prove isn't a way to prove things, it's a way to get confused!**

Here's how you turn a right-side left-side proof into a real proof by using the same simplification steps. Now we start with what we know and deduce the thing we wish to prove. Try to see how the same steps from 'proof' #1 are used in this argument:

A real proof. First, we simplify the left side:

$$\frac{a-b}{a} = \frac{a-b}{a} + \frac{b}{a} - \frac{b}{a} = \frac{a-b+b}{a} - \frac{b}{a} = \frac{a}{a} - \frac{b}{a} = 1 - \frac{b}{a}$$

and the right side:

$$\frac{qr}{rq} - \frac{b}{a} = \frac{qr}{qr} - \frac{b}{a} = 1 - \frac{b}{a}$$

Thus, the two expressions are equal, and we have shown that

$$\frac{a-b}{a} = \left(\frac{q}{r}\right) \left(\frac{r}{q}\right) - \frac{b}{a}$$

□

Right-side left-side has nothing to do with right and left sides. Two sides are ok, as long as you *start with what you know*. Here's a perfectly good, if kind of strange proof (it's strange because each step is hard to predict). It's just the first proof, turned upside down!

Another real proof. We know that $1 = 1$, so we can desimplify this equation:

$$\begin{aligned}
 1 &= 1 && \text{this is self evident!} \\
 \frac{a}{a} &= \frac{qr}{qr} && \text{de-simplify} \\
 \frac{a-b+b}{a} &= \frac{qr}{rq} && \text{de-simplify} \\
 \frac{a-b}{a} + \frac{b}{a} &= \left(\frac{q}{r}\right) \left(\frac{r}{q}\right) && \text{de-simplify} \\
 \frac{a-b}{a} &= \left(\frac{q}{r}\right) \left(\frac{r}{q}\right) - \frac{b}{a} && \text{add something to both sides} \\
 \frac{a-b}{a} &= \left(\frac{q}{r}\right) \left(\frac{r}{q}\right) - \frac{b}{a} && \text{this is what we want}
 \end{aligned}$$

□

Try doing that with the right-side left-side proof of $2 = 3$ and you'll see what's wrong with it!

A nonsense no-good non-proof that $2 = 3$.

$$\begin{aligned}
 0 &= 0 && \text{this is self evident!} \\
 0 \times 2 &= 0 \times 3 && \text{de-simplify} \\
 2 &= 3 && \text{Uh-oh! Can we really cancel zeroes?}
 \end{aligned}$$

□

Frankly, when you have an equation to prove, usually the easiest thing to do is just to rewrite the equation in the form $STUFF = 0$ and try to simplify the $STUFF$ to get the 0. In this case:

The easiest proof.

$$\begin{aligned}
 &\frac{a-b}{a} - \left(\frac{q}{r}\right) \left(\frac{r}{q}\right) + \frac{b}{a} \\
 &= \frac{a-b}{a} - 1 + \frac{b}{a} && \text{simplify} \\
 &= \frac{a-b+b}{a} - 1 && \text{simplify} \\
 &= \frac{a}{a} - 1 && \text{simplify} \\
 &= 1 - 1 && \text{simplify} \\
 &= 0 && \text{simplify}
 \end{aligned}$$

□