1 The spin homomorphism $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{1,3}(\mathbb{R})$

A summary from multiple sources
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Abstract

We will discuss the spin homomorphism $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{1,3}(\mathbb{R})$ in three manners. Firstly we interpret $\mathrm{SL}_2(\mathbb{C})$ as acting on the Minkowski spacetime $\mathbb{R}^{1,3}$; secondly by viewing the quadratic form as a twisted $\mathbb{P}^1 \times \mathbb{P}^1$; and finally using Clifford groups.

1.1 Introduction

The spin homomorphism

$$\mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{1,3}(\mathbb{R})$$

is a homomorphism of classical matrix Lie groups. The lefthand group consists of 2×2 complex matrices with determinant 1. The righthand group consists of 4×4 real matrices with determinant 1 which preserve some fixed real quadratic form Q of signature $(1, 3)$. This map is alternately called the *spinor map* and variations. The image of this map is the identity component of $\mathrm{SO}_{1,3}(\mathbb{R})$, denoted $\mathrm{SO}^+_{1,3}(\mathbb{R})$. The kernel is $\{\pm I\}$. Therefore, we obtain an isomorphism

$$\mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\pm I \simeq \mathrm{SO}^+_{1,3}(\mathbb{R}).$$

This is one of a family of isomorphisms of Lie groups called *exceptional isomorphisms*. In Section 1.3, we give the spin homomorphism explicitly, although these formulae are unenlightening by themselves. In Section 1.4 we describe $\mathrm{O}_{1,3}(\mathbb{R})$ in greater detail as the group of Lorentz transformations.

This document describes this homomorphism from three distinct perspectives. The first is very concrete, and constructs, using the language of Minkowski space, Lorentz transformations and Hermitian matrices, an explicit action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{R}^4$ preserving Q (Section 1.5). The second is geometric, and describes the zero locus of Q as a twisted form of $\mathbb{P}^1 \times \mathbb{P}^1$; it then inherits an action from $\mathbb{P}^1 \times \mathbb{P}^1$ (Section 1.6). The third is the most general and develops the basic theory of Clifford algebras and Clifford groups in order to define the Spin group, before specializing to our case, in which $\mathrm{SL}_2(\mathbb{C}) \simeq \mathrm{Spin}_{1,3}(\mathbb{R})$ (Sections 1.7 to 1.9). This last perspective is the most challenging, but it also leads to other exceptional isomorphisms in Lie theory (which is beyond the scope of this note).

1.2 Preliminaries and Notational conventions

When a matrix Lie group appears in what follows, we will indicate the field over which we are considering its points, e.g. $\mathrm{SL}_2(\mathbb{R})$ vs. $\mathrm{SL}_2(\mathbb{C})$. The exception is the common convention that for the orthogonal group (or special orthogonal group) associated to a quadratic form Q of signature (p, q) on *real* vector space we sometimes write $\mathrm{O}(p, q)$ instead of $\mathrm{O}_{p,q}(\mathbb{R})$.

All the quadratic forms we consider are *non-degenerate*.

1.3 Explicit homomorphism

Fix the quadratic form

$$Q(t, x, y, z) = t^2 - x^2 - y^2 - z^2,$$

which has signature $(1, 3)$ (see Section 1.4 for the definition of signature). Consider $\mathrm{O}_{1,3}(\mathbb{R})$ to be 4×4 real matrices which preserve the form Q. Then the spin homomorphism, given explicitly, is:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto$$
$$\frac{1}{2}\begin{pmatrix} a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d} & a\bar{b} + \bar{a}b + \bar{c}d + c\bar{d} & i(\bar{a}b - a\bar{b} + \bar{c}d - c\bar{d}) & a\bar{a} - b\bar{b} + c\bar{c} - d\bar{d} \\ a\bar{c} + \bar{a}c + b\bar{d} + \bar{b}d & a\bar{d} + \bar{a}d + b\bar{c} + \bar{b}c & i(\bar{a}d - a\bar{d} + b\bar{c} - c\bar{b}) & a\bar{c} + \bar{a}c - b\bar{d} - \bar{b}d \\ i(a\bar{c} - \bar{a}c + b\bar{d} - \bar{b}d) & i(a\bar{d} - \bar{a}d + b\bar{c} - \bar{b}c) & \bar{a}d + a\bar{d} - b\bar{c} - c\bar{b} & i(a\bar{c} - \bar{a}c - b\bar{d} + \bar{b}d) \\ a\bar{a} + b\bar{b} - c\bar{c} - d\bar{d} & a\bar{b} + \bar{a}b - \bar{c}d - c\bar{d} & i(\bar{a}b - a\bar{b} + c\bar{d} - \bar{c}d) & a\bar{a} - b\bar{b} - c\bar{c} + d\bar{d} \end{pmatrix}. \quad (1)$$

1.4 Lorentz group

1.4.1 The orthogonal group $\mathrm{O}(p, q)$

Let E be a vector space of dimension n, over $\mathbb{R}$. By $\mathrm{GL}(E)$ we mean the general linear group on E, i.e. the group of invertible linear transformations. Specifying a quadratic form Q on E is equivalent to specifying a symmetric bilinear form B on E. Since we work over $\mathbb{R}$, such a choice is determined up to a change of basis by the *signature* (p, q), as follows. Fixing a basis, we can represent B by a matrix G, i.e. $B(u, v) = u^T G v$. It is always possible to choose a basis so that G is diagonal with ± 1's on the diagonal; the number p of 1's and q of -1's is an invariant of the form and is called the signature.

Fix such a choice of signature (p, q), where p and q are positive integers such that $p + q = n$ (so Q and B are nondegenerate). The orthogonal group

$O(p,q)$ is the subgroup of $GL(E)$ consisting of all linear transformations preserving Q, or, equivalently, preserving B.

If a basis for E is chosen under which the matrix representation of B is G, then $GL(E)$ and $O(p,q)$ can be given as matrix groups:

$$GL(E) = \{A \in \mathrm{Mat}_{n\times n}(E) : \det(A) \neq 0\},$$

$$O(p,q) = \{A \in GL(E) : A^T G A = G\}.$$

One sees that $O(p,q)$ is determined up to isomorphism by the signature (p,q) alone, justifying the notation. Furthermore, interchanging p and q replaces B with its negative, so that $O(p,q) \cong O(q,p)$.

When p or q is zero, we recover the usual orthogonal group $O(n)$. When p and q are both strictly positive, $O(p,q)$ is called an *indefinite orthogonal group*.

1.4.2 The Lorentz group

A nice book on Minkowski spacetime and Lorentz group is [Nar].

The Lorentz group $O(1,3)$ is one of the most important groups in physics. The relevant four-dimensional vector space is called *Minkowski spacetime* and denoted $\mathbb{R}^{1,3}$ when it is equipped with the quadratic form

$$Q(t,x,y,z) = t^2 - x^2 - y^2 - z^2.$$

Here, t is the coordinate of time and (x,y,z) are the coordinates of space[1]. A vector v in Minkowski spacetime is called *timelike* if $Q(v) < 0$ and *spacelike* if $Q(v) > 0$; if $Q(v) = 0$, it is called *null* or *lightlike*. The null vectors form the *light cone*, the 'inside' of which consists of two components made up of the timelike vectors. If we specify a direction of time, we can refer to these as the *future-pointing* and *past-pointing* timelike vectors. The elements of $O(1,3)$ are called *Lorentz transformations* and by definition, they take timelike vectors to timelike vectors, spacelike to spacelike, and null to null.

The Lorentz group is a real Lie group of dimension six. As Lie groups, the indefinite orthogonal groups are smooth real manifolds. Each member of the family has four connected components. In the case of the Lorentz group, these four components can be characterised by which of the following properties its elements possess (each property is constant on a component):

[1] This quadratic form is the *metric tensor* of Minkowski spacetime

(i) The element preserves (reverses) the direction of time, in which case it is called *orthochronous* (*non-orthochronous*)[2].

(ii) The element preserves (reverses) the orientation of a basis for Minkowski space, in which case it is called *proper* (*improper*).

The subgroup of orthochronous transforms is denoted by $O^+(1,3)$. The subgroup of proper Lorentz transforms is denoted by $SO(1,3)$ (proper elements have determinant 1). The subgroup of all Lorentz transforms preserving both the direction of time and orientation is called the *restricted* Lorentz group and denoted by $SO^+(1,3)$. This is the identity component of the Lorentz group: $SO^+(1,3) = O(1,3)^\circ$. The component group $O(1,3)/SO^+(1,3)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, i.e., the Klein four-group.

1.5 The spin homomorphism via Hermitian matrices

1.5.1 Overview

We shall construct a surjective homomorphism of Lie groups from $SL_2(\mathbb{C})$ to $SO^+(1,3)$. We will show that this homomorphism is two-to-one. Since $SL_2(\mathbb{C})$ is simply connected we deduce that $SL_2(\mathbb{C})$ is the double cover of $SO^+(1,3)$. Composing with the embedding $SO^+(1,3) \hookrightarrow SO(1,3)$ we obtain the desired *spin homomorphism*. For a reference on this material, see [CSM, Lie Groups, Chapter 2] and [Rab].

1.5.2 Minkowski space as the collection of Hermitian matrices

Let $\mathbb{W}$ represent the collection of 2×2 Hermitian matrices,

$$\mathbb{W} = \{W \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : W = W^\dagger\},$$

where $W^\dagger$ denotes the complex conjugate transpose of W.

This can be considered an alternate form of $\mathbb{R}^{1,3}$ via the following map:

$$\mathbb{R}^{1,3} \to \mathbb{W}, \quad (t,x,y,z) \mapsto W = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}.$$

It is convenient that the quadratic form on $\mathbb{R}^{1,3}$ becomes the determinant: $\det W = t^2 - x^2 - y^2 - z^2$.

[2] In other words, a non-orthochronous transformation interchanges future-pointing and past-pointing timelike vectors.

The standard basis for $\mathbb{R}^{1,3}$ is now written for $\mathbb{W}$ in the following way:

$$\begin{aligned} \sigma_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{2}$$

The matrices $\sigma_1, \sigma_2, \sigma_3$ are called the *Pauli matrices*. With this basis, a point (t, x, y, z) in Minkowski spacetime corresponds to the 2×2 matrix $t\sigma_0 + x\sigma_1 + y\sigma_2 + z\sigma_3$. If one defines an inner product by

$$\langle M, N \rangle = \frac{1}{2} \operatorname{Tr}(MN),$$

then one readily checks that this basis is orthonormal:

$$\langle \sigma_\mu, \sigma_\nu \rangle = \delta_{\mu\nu}. \tag{3}$$

This gives us an easy way to write down the inverse map from $\mathbb{W}$ to $\mathbb{R}^{1,3}$:

$$W \mapsto (\langle W, \sigma_0 \rangle, \langle W, \sigma_1 \rangle, \langle W, \sigma_2 \rangle, \langle W, \sigma_3 \rangle)$$

1.5.3 The spin homomorphism

Now we can let $X \in \mathrm{SL}_2(\mathbb{C})$ act on $\mathbb{W}$ via

$$W \mapsto W^X = XWX^\dagger$$

This action preserves the determinant and hence is an action by linear isometry on Minkowski spacetime, by the isomorphism of the last section. Therefore it maps to a subgroup of the Lorentz group:

$$\mathrm{SL}_2(\mathbb{C}) \to \mathrm{O}(1,3)$$

Furthermore, since $\mathrm{SL}_2(\mathbb{C})$ is simply connected as a manifold (as it is a matrix Lie group), and the homomorphism is continuous, we deduce that the image is the full identity component of the Lorentz group.

Therefore we get the promised spin homomorphism

$$\psi : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}^+(1,3), \quad X \mapsto Y. \tag{4}$$

Tracing back through the definitions, one finds that

$$Y_{\mu\nu} = \langle \sigma_\mu, X\sigma_\nu X^\dagger \rangle, \quad \mu, \nu = 0, 1, 2, 3. \tag{5}$$

Example 1. *Consider the element*

$$X = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$$

of $\mathrm{SL}_2(\mathbb{C})$*. By the formula* (5) *above we see that the spin homomorphism sends* X *to the restricted Lorentz transform*

$$Y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Compare to (1)*.*

1.5.4 The kernel of the spin homomorphism

The kernel of the spin homomorphism consists of the matrices $X \in \mathrm{SL}_2(\mathbb{C})$ such that $XWX^\dagger = W$ for all Hermitian W. In particular taking W as the identity matrix we have $XX^\dagger = I$. Thus X is unitary and the action can be rewritten as $W \mapsto XWX^{-1}$. Therefore the kernel of the spin homomorphism is

$$\{X \in \mathrm{SL}_2(\mathbb{C}) : WX = XW \text{ for all Hermitian } W\}.$$

All real diagonal matrices are contained in $\mathbb{W}$. Therefore such X must be diagonal. Finally the condition $\det X = 1$ forces that $X = \pm I$.

Therefore we have an isomorphism

$$\mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{\pm I\} \simeq \mathrm{SO}^+(1,3).$$

This also shows $\mathrm{SL}_2(\mathbb{C})$ is a double cover of $\mathrm{SO}^+(1,3)$. In fact, it is the universal cover, recalling that $\mathrm{SL}_2(\mathbb{C})$ is simply connected.

1.6 A geometric perspective

1.6.1 Overview

We will describe the spin homomorphism as a composition of the natural homomorphisms

$$\mathrm{SL}_2(\mathbb{C}) \hookrightarrow \mathrm{GL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$$

and a homomorphism

$$\mathrm{PGL}_2(\mathbb{C}) \to \mathrm{O}_{1,3}(\mathbb{R})$$

arising from a geometric perspective due to van der Waerden. Our exposition follows [EGM, pp.19–20].

1.6.2 The map $\mathrm{PGL}_2(\mathbb{C}) \to \mathrm{O}_{1,3}(\mathbb{R})$

Choose the quadratic form

$$Q(x_1, x_2, x_3, x_4) = x_1x_2 - x_3^2 - x_4^2$$

which is of signature $(1,3)$ over $\mathbb{R}$, hence equivalent to the usual form giving Minkowski space. Thus $\mathrm{O}_Q(\mathbb{R}) \simeq \mathrm{O}_{1,3}(\mathbb{R})$. We will see in what follows that its projective zero locus is a twisted form of $\mathbb{P}^1 \times \mathbb{P}^1$.

A change of variables

$$(x_1, x_2, x_3, x_4) = A(y_1, y_2, y_3, y_4) := (y_1, y_2, y_3 + iy_4, y_3 - iy_4)$$

replaces Q with a form

$$Q'(y_1, y_2, y_3, y_4) = y_1y_2 - y_3y_4$$

which is equivalent to Q over $\mathbb{C}$ (but not over $\mathbb{R}$). In fact, the projective zero locus of this form is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ over $\mathbb{R}$.

Define a map

$$T : \mathbb{C}^4 \to \mathbb{C}^4, \quad T(\lambda_1, \lambda_2, \mu_1, \mu_2) \mapsto (\lambda_1\mu_1, \lambda_2\mu_2, \lambda_1\mu_2, \lambda_2\mu_1).$$

This map parametrizes the affine quadric $Q'(y_1, y_2, y_3, y_4) = 0$. Projectivizing, we have a quadric in $\mathbb{P}^3$, and fixing λ_1/λ_2 or μ_1/μ_2 parametrizes the quadric as a union of projective lines; in other words, the projective quadric is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ over $\mathbb{R}$, i.e.

$$T : \mathbb{P}^1 \times \mathbb{P}^1 \to \{[y_1, y_2, y_3, y_4] \in \mathbb{P}^3 : Q'(y_1, y_2, y_3, y_4) = 0\}$$

is an isomorphism. Since PGL_2 forms the automorphism group on $\mathbb{P}^1$, it is natural to define the following action by $g \in \mathrm{PGL}_2(\mathbb{C})$ on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$:

$$g([\lambda_1, \lambda_2], [\mu_1, \mu_2]) = ([a\lambda_1 + b\lambda_2, c\lambda_1 + d\lambda_2], [\overline{a}\mu_1 + \overline{b}\mu_2, \overline{c}\mu_1 + \overline{d}\mu_2]).$$

Here, the complex conjugation ensures that, when tracing back through the twist, this corresponds to an *algebraic* linear action on the zero locus of Q, hence an element of $\mathrm{O}_Q(\mathbb{R})$. Therefore we have defined a map

$$\mathrm{PGL}_2(\mathbb{C}) \to \mathrm{O}_{1,3}(\mathbb{R}).$$

(Notice that $\mathrm{PGL}_2(\mathbb{C})$ is isomorphic to $\mathrm{PSL}_2(\mathbb{C})$).

1.7 Clifford algebras

1.7.1 Overview

The Clifford algebra associated to a quadratic form Q on a vector space E is the universal K-algebra such that E can be embedded by a linear map and squaring in the image of E corresponds to the quadratic form Q. This brings into play plenty of extra structure beyond the quadratic space E. This section was written with the aid of [Por, Lan, EGM].

1.7.2 Tensor algebras

We state the definition in terms of a module over a commutative ring, but we will use it for a vector space over a field.

Let M be a module over a commutative ring A. The *tensor algebra* $T(M)$ (also called a *free algebra*) is a graded object:

$$T(M) = \oplus_{r=0}^{\infty} T^r(M)$$

where the gradings are:

$$T^0(M) = A, \quad T^r(M) = M \otimes_A M \otimes_A M \cdots \otimes_A M$$

where the tensor is taken r times. It is equipped with a multiplication that respects the grading given by associativity of tensor products, for example

$$(x_1 \otimes x_2)(x_3 \otimes x_4 \otimes x_5) = (x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5).$$

If $a \in T^0(M)$ and $v \in T^r(M)$, then $a \cdot v$ is just given by the scalar multiplication on the tensor product. That is, the multiplication

$$T^r(M) \times T^s(M) \to T^{r+s}(M)$$

is associative and bilinear with respect to A, so that $T(M)$ becomes a ring. In fact, it is an A-algebra, in the sense that A embeds into $T(M)$ as the 0-th grade. Note that M is mapped into the 1-st grade. We say elements in $T^r(M)$ are *homogeneous* of degree r.

Taking the tensor algebra is functorial: module maps $f : M \to N$ give tensor algebra maps $T(f) : T(M) \to T(N)$. The map $T(f)$ acts on each coordinate in a simple tensor.

1.7.3 Definition of Clifford algebras

In the rest of this expository article we let K be a field of characteristic not equal to 2 and assume that E is a finite dimensional vector space over K with a *nondegenerate* quadratic form Q on it.

A Clifford algebra $C_Q = C_Q(K)$ associated to the quadratic form Q on the vector space E is a K-algebra with the properties that

(i) We can embed E in C_Q by a linear map.

(ii) Squaring in $E \subset C_Q$ corresponds to the quadratic form Q.

(It is in fact the universal such object; see [Lan, XIX §4.]). It is formed as a quotient of the tensor algebra $T(E)$ (Section 1.7.2) (so that item (i) is satisfied). Let Φ_Q be the symmetric bilinear form associated to a quadratic form Q, namely

$$\Phi_Q(x,y) = \frac{1}{2}\big(Q(x+y) - Q(x) - Q(y)\big).$$

The Clifford algebra is

$$C(Q) := T(E)/\mathfrak{a}_Q$$

where $\mathfrak{a}_Q$ is the ideal generated by the elements

$$x \otimes y + y \otimes x - 2\Phi_Q(x,y), \quad x, y \in E.$$

This implies that in $C(Q)$, we have

$$x \cdot x = Q(x).$$

Please note that in general, however, $x \cdot y \neq \Phi_Q(x,y)$ (in other words, the multiplication is determined by Φ_Q but is not just given by Φ_Q directly). In particular, the multiplication in $C(Q)$ is not in general commutative. The Clifford algebra is, of course, no longer graded by $\mathbb{Z}$, but we will see later that it is $\mathbb{Z}/2\mathbb{Z}$-graded.

1.7.4 Basis and computations in a Clifford algebra

We will now specify a basis of C_Q as a K-vector space. Choose an orthonormal basis $(e_1, \ldots, e_n)$ of E with respect to Φ_Q. In C_Q we can multiply elements of E, and we find that

$$e_i \cdot e_j = -e_j \cdot e_i, \quad \text{for } i \neq j, \quad \text{and} \quad e_i^2 = Q(e_i). \tag{6}$$

Let $\mathcal{P}_n$ be the power set of $\{1, 2, \ldots, n\}$. It will index the basis. For $M \in \mathcal{P}_n$ consisting of elements $j_1 < \cdots < j_r$, define

$$e_M := e_{j_1} \cdots e_{j_r},$$

and $e_\emptyset := 1$. Then a basis of C_Q is

$$\{e_M : M \in \mathcal{P}_n\}.$$

(See [BtD] Chapter I, Corollary 6.7). In particular, C_Q is a 2^n-dimensional vector space over K, where $n = \dim_K E$. By the relations (6) above, the product of two basis vectors e_M and e_N is a K-multiple of some other basis vector e_L.

1.7.5 Examples

Let $K = \mathbb{R}$ and $E = \mathbb{R}^n$. Every nondegenerate quadratic form Q on E is equivalent to one of signature (p, q). The corresponding Clifford algebra is denoted by $C_{p,q}(\mathbb{R})$.

We can check that $C_{0,0}(\mathbb{R}) \simeq \mathbb{R}$ and $C_{0,1}(\mathbb{R}) \simeq \mathbb{C}$. The algebra $C_{0,2}(\mathbb{R})$ has $\mathbb{R}$-basis $(1, e_1, e_2, e_1e_2)$ where (e_1, e_2) is the standard basis of $\mathbb{R}^2$. Direct computation shows that the last three basis elements square to -1 and anti-commute, so we recover the Hamilton quaternions.

If $K = \mathbb{C}$, every nondegenerate quadratic form is equivalent to the diagonal form

$$Q(z) = z_1^2 + \cdots + z_n^2.$$

Therefore, up to isomorphism there is a unique nondegenerate Clifford algebra $C_n(\mathbb{C})$ for each n. The first few examples are $C_0(\mathbb{C}) \simeq \mathbb{C}$, $C_1(\mathbb{C}) \simeq \mathbb{C} \oplus \mathbb{C}$, and $C_2(\mathbb{C}) \simeq \mathrm{Mat}_{2\times 2}(\mathbb{C})$, the algebra of 2×2-complex matrices.

On the other hand, if the quadratic form Q is zero, then the Clifford algebra is isomorphic to the exterior algebra $\bigwedge(E) = \bigoplus_{j=0}^{\dim_K E} \bigwedge^j(E)$.

1.7.6 The main involution and the decomposition of the Clifford algebra

An *automorphism* of an algebra is an invertible linear map which preserves multiplication. An *anti-automorphism* is an invertible linear map which reverses multiplication, i.e. $f(ab) = f(b)f(a)$. An automorphism or anti-automorphism is an *involution* if it is self-inverse.

The reflection through the origin $x \mapsto -x$ on E is an involution that extends to a linear map (denoted *) on the tensor algebra $T(E)$ by acting coordinatewise on tensor products, i.e.

$$(x_1 \otimes \cdots \otimes x_k)^* = -x_1 \otimes \cdots \otimes -x_k.$$

This map on $T(E)$ preserves the ideal $\mathfrak{a}_Q$ defining C_Q and so it descends to a map on C_Q. This is called the *main involution* or *grade involution*. Let $M \in \mathcal{P}_n$ with $|M| = r$. Then by (6), we have

$$e_M^* := (-1)^r \cdot e_M.$$

We have a decomposition of C_Q introduced by the main involution *: $C_Q = C_Q^0 \oplus C_Q^1$ where $C_Q^j = \{x \in C_Q \mid x^* = (-1)^j x\}$. And because * is an automorphism we have

$$C_Q^i C_Q^j = C_Q^{i+j}$$

where $i, j \in \mathbb{Z}/2\mathbb{Z}$. This gives the $\mathbb{Z}/2\mathbb{Z}$-grading structure on C_Q. The subspace C_Q^0 is a subalgebra of C_Q, called the even subalgebra[3]. The elements which lie completely in either C_Q^0 or C_Q^1 are called *homogeneous*.

1.7.7 The main anti-involution

A tensor algebra $T(E)$ has a unique anti-involution called the *main anti-involution* or *canonical anti-involution* or *transposition*:

$$(x_1 \otimes \cdots \otimes x_k)^t := x_k \otimes \cdots \otimes x_1.$$

It is the identity on E (the 1-st grading) and it preserves the ideal $\mathfrak{a}_Q$ defining the Clifford algebra, so that it descends to an anti-involution on the Clifford algebra of the same name. Using (6), we obtain

$$e_M^t = (-1)^{\frac{r(r-1)}{2}} \cdot e_M.$$

We observe that these two involutions * and ^{t} commute. One can also construct an anti-involution $\overline{x} := x^{*t}$ which must then satisfy

$$\overline{e_M} = (-1)^{\frac{r(r+1)}{2}} \cdot e_M.$$

[3]The odd part C_Q^1 is not a subalgebra.

1.7.8 The Clifford norm and spinor norm

Using the involutions we can define two important norms in C_Q.

For any $x \in C_Q$ we define the *Clifford norm* as

$$\mathrm{N}(x) = x\overline{x} = \overline{x}x$$

and the *spinor norm* as

$$Q(x) = x^t x = x x^t.$$

Both of these norms are homomorphisms from C_Q to K. We use the notation Q for the spinor norm because restricted to the anisotropic vectors of Q in E (those $v \in E$ such that $Q(v) \neq 0$) the spinor norm coincides with the quadratic form Q.

1.8 Clifford group

1.8.1 Overview

The Clifford group Γ is a subgroup of the unit group of the Clifford algebra, and it comes with a natural action on E, i.e. a representation

$$\rho : \Gamma \to \mathrm{GL}(E).$$

We will see that the elements of the Clifford group from E act by reflection in the corresponding hyperplane, and these elements generate the full orthogonal group $\mathrm{O}_Q(K)$ by the Cartan-Dieudonné Theorem; this is in fact the full image. The sequence of reasoning and proofs in this section are all adapted from [BtD, Chapter 1.6].

1.8.2 Definition of the Clifford group

We will write $C_Q^\times$ for the unit group of the Clifford algebra. Following Atiyah, Bott and Shapiro's convention, we define the *twisted conjugation* action on a Clifford algebra by an invertible element x as

$$y \mapsto y^x = x^* y x^{-1}.$$

The *Clifford group* is:

$$\Gamma = \Gamma_Q(K) = \{x \in C_Q^\times : x \text{ is homogeneous and } x^* E x^{-1} \subset E\}.$$

The main involution is an automorphism of C_Q, therefore Γ is a group.

Clearly, $K^\times \subset \Gamma$. Furthermore, the anisotropic elements of E are in $C_Q^\times$ and therefore in Γ.

We will divide Γ into two complementary subsets:

$$\Gamma^0 = \Gamma \cap C_Q^0, \quad \Gamma^1 = \Gamma \setminus \Gamma^0.$$

In other words, Γ^j is the subset of elements of degree j ($j = 0, 1$). Moreover Γ^0 is a normal subgroup of Γ of index 2. Elements of Γ^0 will be called *even*, while elements of Γ^1 will be called *odd*. Every element of Γ can be written as a sum of an even and an odd.

1.8.3 The representation $\rho : \Gamma \twoheadrightarrow \mathrm{O}_Q(K)$

The *natural representation* of Γ on E is

$$\rho : \Gamma \to \mathrm{GL}(E), \quad x \mapsto \rho_x$$

given by

$$\rho_x : E \to E, \quad v \mapsto x^* v x^{-1}.$$

Our goal is to prove the following theorem.

Theorem 2. *The image $\rho(\Gamma)$ is equal to the orthogonal group $\mathrm{O}_Q(K)$.*

We will need a sequence of lemmas.

Lemma 3. *For any $x \in E \subset C_Q$ with $Q(x) \neq 0$, the map $\rho_x : E \to E$ given by $\rho_x(v) = x^* v x^{-1}$ is the reflection about the hyperplane $x^\perp$ orthogonal to the vector x.*

Proof. Recall that the reflection s about the hyperplane $x^\perp$ orthogonal to the vector x is given by

$$s(v) = v - 2\frac{\Phi_Q(v, x)}{Q(x)} x.$$

But we are computing within the Clifford algebra and hence $x^2 = Q(x) \cdot 1$ and $2\Phi_Q(v, x) \cdot 1 = vx + xv$.

Therefore we compute that

$$\begin{aligned}
s(v) &= v - 2\frac{\Phi_Q(v,x)}{Q(x)}x \\
&= v - 2\Phi_Q(v,x)\frac{x}{Q(x)} \\
&= v - (vx + xv)x^{-1} \\
&= -xvx^{-1} \\
&= x^* v x^{-1}
\end{aligned}$$

since for all $x \in E$ the main involution gives $x^* = -x$. □

Lemma 4. *The kernel of the homomorphism $\rho : \Gamma \to \mathrm{GL}(E)$ is $K^\times$.*

Proof. The kernel clearly contains $K^\times$. Conversely, consider an element of Γ that acts trivially; write it a_0+a_1, with a_0 even and a_1 odd. Then $(a_0+a_1)v = v(a_0 + a_1)^* = v(a_0 - a_1)$ for all $v \in E$. Separating even and odd parts we deduce that $a_0 v = v a_0$ and $a_1 v = -v a_1$. Now choose an orthonormal basis $\epsilon_1, \dots, \epsilon_n$ for E with respect to Q. We may write $a_0 = x + \epsilon_1 y$ where $x \in C_Q^0$ and $y \in C_Q^1$ and neither x nor y contains a factor of ϵ_1, so $\epsilon_1 x = x \epsilon_1$ and $\epsilon_1 y = -y \epsilon_1$. Using the relation $a_0 v = v a_0$ with $v = \epsilon_1$ we obtain that $y = 0$. Thus a_0 has no monomials with a factor ϵ_1. Similarly, it has no monomials with any ϵ_i, meaning $a_0 \in K$.

Similarly, write $a_1 = y + \epsilon_1 x$ with x and y not containing a factor of ϵ_1. The relation $a_1 \epsilon_1 = -\epsilon_1 a_1$ tells us that $x = 0$. Repeating with the other basis elements, we deduce that $a_1 = 0$.

Therefore $a_0 + a_1 = a_0 \in K \cap \Gamma = K^\times$. □

Lemma 5. *Restricting the spinor norm Q to Γ gives a homomorphism $Q : \Gamma \to K^\times$.*

Proof. For $x \in \Gamma$, to show that $Q(x) = x^t x \in K^\times$, the key idea is to show that $\rho(x^t x) = 1$ and then use Lemma 4.

We firstly show that if $x \in \Gamma$ then $\mathrm{N}(x)$ acts trivially on E. Since $x \in \Gamma$, we have $x^* v x^{-1} \in E$ for all $v \in E$. Moreover, transpose $v \mapsto v^t$ is the identity on E. We compute

$$\begin{aligned}(\overline{x}x)^* v(\overline{x}x)^{-1} &= x^t x^* v\left(x^{*t}x\right)^{-1} \\ &= x^t x^* v x^{-1}\left(x^{*t}\right)^{-1} \\ &= x^t (x^* v x^{-1})^t \left(x^{*t}\right)^{-1} \\ &= x^t (x^{-1})^t v x^{*t}(x^{*t})^{-1} \\ &= v\end{aligned}$$

Thus $\overline{x}x = \mathrm{N}(x)$ is in the kernel of ρ. Also notice that $\mathrm{N} : \Gamma \to K^\times$ is a homomorphism because for all $x, y \in \Gamma$ we have $\overline{xy} = \overline{y}\,\overline{x}$.

Recall that Γ consists of homogeneous elements. On the even elements of Γ the spinor norm Q agrees with the Clifford norm N, and on the odd elements $Q = -\mathrm{N}$. Therefore Q is also a homomorphism from Γ to $K^\times$ by Lemma 4.

□

Proof of Theorem 2. By Lemma 5, Q is a homomorphism on Γ. Therefore,

$$Q(x^* v x^{-1}) = Q(x^*)Q(v)Q(x)^{-1} = Q(v)$$

and therefore the action on E preserves the quadratic form Q of E. Thus we have a homomorphism $\Gamma \to \mathrm{O}_Q(K)$. We have shown in Lemma 3 that any $x \in E \subset \Gamma$ acts on E by reflection through the hyperplane $x^\perp$. Moreover with the hypothesis that the characteristic of K is not 2, the Cartan-Dieudonné theorem tells us that *every element in* $\mathrm{O}_Q(K)$ *is a product of at most* $\dim E$ *simple reflections* (cf. [Lam, Chapter I, Theorem 7.1]). Hence we deduce that the image of $\rho(\Gamma)$ is $\mathrm{O}_Q(K)$. □

1.9 The spin homomorphism via the Clifford group

1.9.1 Overview

In the Clifford group, we define the Pin and Spin subgroups, which therefore map onto $\mathrm{O}_Q(K)$ and $\mathrm{SO}_Q(K)$. In our case of interest, we show that $\mathrm{Spin}_{1,3}(\mathbb{R})$ is isomorphic to $\mathrm{SL}_2(\mathbb{C})$. This recovers the Spin homomorphism.

1.9.2 Pin and Spin groups

The spinor norm on Γ restricts to the squaring map $x \mapsto x^2$ on $K^\times$, so the elements of K with spinor norm 1 are $\{\pm 1\}$. By Theorem 2, and Lemmas 4 and 5, we get a map induced from the spinor norm which is also denoted by $Q : \mathrm{O}_Q \to K^\times/(K^\times)^2$. It is the unique homomorphism sending reflection through $x^\perp$ to $Q(x)$ modulo $(K^\times)^2$.

From this perspective, it is natural then to define the Pin group:

$$\mathrm{Pin}_Q(K) = \{x \in \Gamma(K) \mid Q(x) = 1\}.$$

Define also

$$\mathrm{Spin}_Q(K) = \mathrm{Pin}_Q(K) \cap \Gamma^0(K),$$

the even elements of Pin_Q. Then we have two exact sequences:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Pin}_Q(K) \longrightarrow \mathrm{O}_Q(K) \longrightarrow K/(K^\times)^2 \tag{7}$$

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_Q(K) \longrightarrow \mathrm{SO}_Q(K) \longrightarrow K/(K^\times)^2$$

1.9.3 Examples, $K = \mathbb{R}$.

Let us see some examples. Let $K = \mathbb{R}$. Then $\mathbb{R}/(\mathbb{R}^\times)^2 = \{\pm 1\}$. Let E be an n-dimensional real vector space.

Positive Definite. If Q is a positive definite quadratic form over $\mathbb{R}$, we write O_n, Spin_n etc. In this case, the spinor norm Q maps everything to 1 (modulo squares). So the spinor norm on positive definite space is trivial. We get double covers

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Pin}_n(\mathbb{R}) \longrightarrow \mathrm{O}_n(\mathbb{R}) \longrightarrow 1$$

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_n(\mathbb{R}) \longrightarrow \mathrm{SO}_n(\mathbb{R}) \longrightarrow 1$$

One can verify that $\mathrm{Spin}_3(\mathbb{R})$ is isomorphic to $\mathrm{SU}(2)$, the group of 2×2 unitary matrices with determinant 1, which is in turn isomorphic to the group of norm 1 Hamilton quaternions. Since $\mathrm{SU}(2)$ is the double cover of $\mathrm{SO}_3(\mathbb{R})$ we have verified the last exact sequence above for $n = 3$.

Negative definite. On the other hand, if Q is negative definite, because every reflection has image -1 in $\mathbb{R}/(\mathbb{R}^\times)^2$, so the spinor norm is the determinant map $\det : \mathrm{O}(\mathbb{R}) \to \{\pm 1\}$.

Indefinite. Now we look at the indefinite signature case. For instance, in the Minkowski spacetime $\mathbb{R}^{1,3}$, reflection through a spacelike vector (space inversion) has spinor norm -1 and $\det -1$; reflection through a timelike vector (time reversal) has spinor norm $+1$ and $\det -1$. We recover the component group of the Lorentz group $\mathrm{O}(1,3)$, defined by $\mathrm{O}(1,3)/\mathrm{SO}^+(1,3)$, which is isomorphic to the Klein four group.

1.9.4 The isomorphism $\mathrm{SL}_2(\mathbb{C}) \simeq \mathrm{Spin}_{1,3}(\mathbb{R})$

We follow [EGM, Chapter 1, 1.3] to construct this isomorphism (and omit details).

Let $(f_j)_{0\le j\le 3}$ be an orthonormal basis of $\mathbb{R}^{1,3}$ associated to the quadratic form

$$Q(tf_0 + xf_1 + yf_2 + zf_3) = Q(t,x,y,z) = t^2 - x^2 - y^2 - z^2.$$

The corresponding Clifford algebra is $C_{1,3}(\mathbb{R})$.

Inside the Minkowski spacetime $\mathbb{R}^{1,3}$ we have two sub-vector spaces $V_0 = \mathbb{R}f_0 + \mathbb{R}f_1 + \mathbb{R}f_2$ and $V' = \mathbb{R}f_3$. On V_0 we have a quadratic form

$$Q_0(tf_0 + xf_1 + yf_2) = Q_0(t,x,y) = t^2 - x^2 - y^2,$$

while on V' we have another quadratic form $Q'(ze_3) = -z^2$. The Clifford algebra associated to (V', Q') (denoted by $C_{0,1}(\mathbb{R})$) is isomorphic to $\mathbb{C}$.

We firstly describe the isomorphism $\mathrm{Mat}_{2\times 2}(\mathbb{R}) \simeq C^0_{1,2}(\mathbb{R})$. To do this we define

$$\begin{aligned} \tau_0 &= \frac{1}{2}(f_0 + f_1), \qquad \tau_1 = \frac{1}{2}(f_0 - f_1), \\ u &= \tau_1\tau_0 = \frac{1}{2}(1 + f_0f_1),\ w_1 = \tau_1 f_2 = \frac{1}{2}(f_0f_2 - f_1f_2), \\ w_0 &= \tau_0 f_2 = \frac{1}{2}(f_0f_2 + f_1f_2),\ v = \tau_0 f_1 = \frac{1}{2}(1 - f_0f_1). \end{aligned} \tag{8}$$

We claim that

Proposition 6. *[EGM, Chapter 1, Proposition 3.2] The correspondences*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mapsto u, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto w_1, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto w_0, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mapsto v$$

extends to an algebra isomorphism $\psi : \mathrm{Mat}_{2\times 2}(\mathbb{R}) \to C^0_{1,2}(\mathbb{R})$.

Next, we define a map $^\bullet : C_{0,1}(\mathbb{R}) \to C^0_{1,3}(\mathbb{R})$ by the following formula:

$$\Big(\sum_{M\in\mathcal{P}_n} \lambda_M e_M\Big)^\bullet = \sum_{M\in\mathcal{P}_n} \lambda_M (f_0 f_1 f_2)^{\epsilon_M} e_M$$

where ϵ_M is 0 or 1 if the cardinality of M is even or odd. One can show that this map $^\bullet$ commutes with the transpose ^{t} of Clifford algebras. Moreover the map $^\bullet : C_{0,1}(\mathbb{R}) \to C^0_{1,3}(\mathbb{R})$ is an $\mathbb{R}$-algebra monomorphism.

Now we combine the $^\bullet$ operation and the isomorphism in Proposition 6 and obtain the following result.

Proposition 7. *[EGM, Chapter 1, Proposition 3.5] The map*

$$\psi : \mathrm{Mat}_{2\times 2}(C_{0,1}(\mathbb{R})) = \mathrm{Mat}_{2\times 2}(\mathbb{C}) \to C^0_{1,3}(\mathbb{R})$$

defined by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \alpha^\bullet u + \beta^\bullet w_1 + \gamma^\bullet w_0 + \delta^\bullet v$$

is an $\mathbb{R}$*-algebra isomorphism and satisfies*

$$\left(\psi\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right)^t = \psi\begin{pmatrix} \delta^t & -\beta^t \\ -\gamma^t & \alpha^t \end{pmatrix}$$

This $\mathbb{R}$*-algebra isomorphism* ψ *restricts to a group isomorphism*

$$\psi : \mathrm{SL}_2(C_{0,1}(\mathbb{R})) = \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Spin}_{1,3}(\mathbb{R}).$$

Remark. More generally, by the structure theory of Clifford algebras, for all $p, q \in \mathbb{Z}_{>0}$ we have the following isomorphisms:

(i) $C_{p+1,q+1}(\mathbb{R}) = \mathrm{Mat}_{2\times 2}(\mathbb{R}) \otimes C_{p,q}(\mathbb{R})$;

(ii) $C^0_{p,q+1}(\mathbb{R}) = C_{p,q}(\mathbb{R})$.

Combine with $C_{0,1}(\mathbb{R}) = \mathbb{C}$ and $\mathrm{Mat}_{2\times 2}(\mathbb{R}) \otimes \mathbb{C} = \mathrm{Mat}_{2\times 2}(\mathbb{C})$, we have the following isomorphism: $\mathrm{Mat}_{2\times 2}(\mathbb{C}) \simeq C_{1,2}(\mathbb{R}) \simeq C^0_{1,3}(\mathbb{R})$. Now we restrict the isomorphism $\psi : \mathrm{Mat}_{2\times 2}(\mathbb{C}) \simeq C^0_{1,3}(\mathbb{R})$ to the norm 1 ($\det 1$) elements and obtain the isomorphism $\psi : \mathrm{SL}_2(\mathbb{C}) \simeq \mathrm{Spin}_{1,3}(\mathbb{R})$.

1.9.5 The Spin homomorphism

Combining (7) with the isomorphism of the last section, we recover the spin homomorphism.

References

[BtD] T. Bröcker, T. tom Dieck, "Representations of compact Lie groups" Graduate texts in mathematics 98, Springer-Verlag, 1985.

[CSM] R. Carter, G. Segal, I. Macdonald, "Lectures on Lie Groups and Lie Algebras" London Mathematical Society Student Texts 32, Cambridge, UK: Cambridge University Press, 1995.

[Con] B. Conrad, "Clifford algebras and Spin groups", handout of the course Math 210C, available at `http://math.stanford.edu/~conrad/210CPage/handouts/clifford.pdf`

[EGM] J. Elstrodt, F. Grunewald, J. Mennicke, "Group Acting on Hyperbolic Space, Harmonic Analysis and Number Theory", Springer-Verlag, 1998.

[Lam] T. Y. Lam, "Introduction to quadratic forms over fields", Graduate Studies in Mathematics 67, Providence, RI: American Mathematical Society, 2005.

[Lan] S. Lang, "Algebra", Third Edition, Reading, MA: Addison-Wesley Publishing Company, 1993.

[Nar] G. L. Naber, "The Geometry of Minkowski Spacetime, An Introduction to the Mathematics of the Special Theory of Relativity", Applied Mathematical Sciences, *Volume* 92, 2nd ed. Springer-Verlag, 2012.

[Por] I. R. Porteous, "Clifford Algebras and the Classical Groups", Cambridge studies in advanced mathematics 50, Cambridge, UK: Cambridge University Press, 1995.

[Rab] J. M. Rabin, "Introduction to Quantum Field Theory for Mathematicians", in the book "Geometry and Quantum Field Theory", IAS/PARK CITY Mathematics Series *Volume* 1, American Mathematical Society, Institute for Advanced Study.

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