Visualizing the arithmetic of imaginary quadratic fields

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Part I: Apollonian Circle Packings
A Descartes quadruple is any collection of four circles which are pairwise mutually tangent, with disjoint interiors.
Given any three mutually tangent circles, there are exactly two ways to complete the triple to a Descartes quadruple.
Beginning with any three mutually tangent circles...
Beginning with any three mutually tangent circles, add in both new circles which would complete the triple to a Descartes quadruple.
Repeat: for every triple of mutually tangent circles in the collection, add the two ‘completions.’
Repeating ad infinitum, we obtain an Apollonian circle packing.
Repeating ad infinitum, we obtain an Apollonian circle packing.
The outer circle has curvature -6 (its interior is outside).
The curvatures (inverse radii) in a Descartes configuration satisfy

\[ 2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2. \]

If \( a, b, c \) are fixed, there are two solutions \( d, d' \), where

\[ d + d' = 2(a + b + c). \]

Hence an integer Descartes quadruple generates an Apollonian packing of integer curvatures.
Conjecture (Graham–Lagarias–Mallows–Wilks–Yan, Fuchs–Sanden)

Let $\mathcal{P}$ be a primitive, integral ACP. Let $S$ be the set of residues of curvatures modulo 24. Then any sufficiently large integer with a residue in $S$ occurs as a curvature.

- Bourgain, Fuchs: Curvatures have positive density in $\mathbb{Z}$.
- Bourgain, Kontorovich: Density one occur.
Apollonian group

\[ \mathcal{A} = \langle S_1, S_2, S_3, S_4 : S_i^2 = 1 \rangle \]

Image from Graham, Lagarias, Mallows, Wilks, Yan
The ‘Superpacking’

Super-Apollonian group

Images from Graham, Lagarias, Mallows, Wilks, Yan
Super-Apollonian group

\[ \langle S_1, S_2, S_3, S_4, S_1^\perp, S_2^\perp, S_3^\perp, S_4^\perp : S_i^2 = (S_i^\perp)^2 = (S_i S_j^\perp)^2 = (S_j^\perp S_i)^2 = 1 \rangle \]
Apollonian generalizations

Apollonian lattices

Thank you to David Wilson and Lionel Levine.
Part II: The Farey subdivision

\[ \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d} \]
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\]
An arborist’s view of $\mathbb{P}^1(\mathbb{Z})$
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Money may not, but matrices do: $\text{SL}_2^+(\mathbb{Z})$
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\[ M \mapsto M \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \]
The address of $\alpha \in \mathbb{R}$

\[
\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}
\]
The address of $\alpha \in \mathbb{R}$

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\[ \frac{a}{b} < \frac{a + c}{b + d} < \frac{c}{d} \]
The address of $\alpha \in \mathbb{R}$

\[
\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}
\]
The address of $\alpha$

Real number $\alpha$

Infinite path through tree:

$$L^{a_0} R^{a_1} L^{a_2} R^{a_3} \ldots$$

Matrix product:

$$\begin{pmatrix} 1 & 0 \\ a_0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_3 \\ 0 & 1 \end{pmatrix} \ldots$$
The Farey subdivision: Continued fractions / Euclidean algorithm

\[ \alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots + \frac{1}{a_n}}}} \]

\[ x_{n+2} = -x_{n+1}a_n + x_n \]

\[ \vdots \]

\[ x_4 = -x_3a_2 + x_2 \]

\[ x_3 = -x_2a_1 + x_1 \]

\[ x_2 = -x_1a_0 + x_0 \]

\[
\begin{pmatrix}
  p_n \\
  q_n
\end{pmatrix}
= \begin{pmatrix}
  0 & 1 \\
  1 & a_0
\end{pmatrix}
\begin{pmatrix}
  0 & 1 \\
  1 & a_1
\end{pmatrix}
\begin{pmatrix}
  0 & 1 \\
  1 & a_2
\end{pmatrix}
\begin{pmatrix}
  0 & 1 \\
  1 & a_3
\end{pmatrix}
\cdots
\begin{pmatrix}
  0 & 1 \\
  1 & a_n
\end{pmatrix}
\begin{pmatrix}
  1 \\
  1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
  1 & 0 \\
  a_0 & 1
\end{pmatrix}
\begin{pmatrix}
  1 & a_1 \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  1 & 0 \\
  a_2 & 1
\end{pmatrix}
\begin{pmatrix}
  1 & a_3 \\
  0 & 1
\end{pmatrix}
\cdots
\begin{pmatrix}
  1 & a_n \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  0 & 1 \\
  1
\end{pmatrix}
\]
Farey subdivision: frothy version

image from Allen Hatcher’s *Topology of Numbers*
Farey subdivision: frothy version

image from Allen Hatcher’s *Topology of Numbers*
Continued fractions as geodesics

address of \( \alpha = L^{a_0} R^{a_1} L^{a_2} R^{a_3} \cdots = [a_0, a_1, a_2, \ldots] \)

Image from Caroline Series’ *The Geometry of Markoff Numbers*
Farey Tessellation

Image of \( \{0, \infty\} \) (and its geodesic) under \( \text{PSL}_2(\mathbb{Z}) \).
Part III: From Integers to Gaussian Integers

- Continued fractions for $\mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{-2}]$ etc.
Schmidt Arrangement of $\mathbb{Q}(i)$
Schmidt Arrangement of $\mathbb{Q}(i)$
Part IV: Quadratic Imaginary Fields

The *Schmidt arrangement* of a imaginary quadratic field $K$ is the orbit of $\hat{\mathbb{R}}$ under the Möbius transformations given by the *Bianchi group*

$$\text{PSL}_2(\mathcal{O}_K) = \left\{ \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} : \alpha, \beta, \delta, \gamma \in \mathcal{O}_K, \alpha \delta - \beta \gamma = 1 \right\} / \pm 1$$

That is,

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \leftrightarrow \left( z \mapsto \frac{\alpha z + \gamma}{\beta z + \delta} \right).$$

Each individual image $M(\hat{\mathbb{R}})$ is called a *$K$-Bianchi circle*.

$$S_K = \{K\text{-Bianchi circles}\}$$
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-7})$
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-2})$
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-11})$
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-19})$
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-5})$
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-6})$
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-15})$
Schmidt Arrangement of $\mathbb{Q}(\sqrt{-3})$

Now the theme on AMS YouTube, Twitter, etc.
Basic properties of $S_K$

$$\Delta = \text{Disc}(K)$$

**Proposition (S.)**

The curvatures in $S_K$ lie in $\sqrt{-\Delta} \mathbb{Z}$.

**Proposition (S.)**

$K$-Bianchi circles intersect at points in $K$, at angles $\theta$ such that $e^{i\theta}$ is a unit in $K$. 

![Diagram of intersecting circles with angle $\theta$]
Circles are ideal classes

\[ M = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \]

**Theorem (S.)**

\[ \left\{ \text{oriented circles} \right\} \Bigg/ \left\{ \text{translations by } \mathcal{O}_K \text{ and rotations by ‘unit angles’} \right\} \]

\[ \uparrow \hspace{1cm} \uparrow \]

\[ \left\{ \text{invertible ideal classes} \right\} \hspace{0.5cm} \left\{ f \in \mathbb{Z}^0, a\mathcal{O}_K \sim \mathcal{O}_K \right\} \hspace{0.5cm} \beta \mathbb{Z} + \delta \mathbb{Z} \]

\[ \uparrow \downarrow \hspace{1cm} \uparrow \downarrow \]

**Corollary:** Number of circles of curvature \( f \) (up to equivalence) is \( h_f/h_K \). (GLMWY for \( \mathbb{Q}(i) \))
Euclideanity and $S_K$

The tangency graph $G_K$ of $S_K$ is:

\[
\begin{cases}
\text{vertices} = \text{circles} \\
\text{edges} = \text{tangencies}
\end{cases}
\]

Proposition (S.)

$G_K$ is connected if and only if $O_K$ is Euclidean.

Proof.

1. Connected component of $\hat{R}$ is all circles reachable by combinations of elementary matrices.
2. Thm of P.M. Cohn: $O_K$ is Euclidean if and only if $\text{SL}_2(O_K)$ is generated by elementary matrices.
Euclideanity and $S_K$

**Theorem (S.)**

$S_K$ is connected if and only if $\mathcal{O}_K$ is Euclidean.

The *ghost circle* is the circle orthogonal to the unit circle having center

\[
\frac{1}{2} + \frac{\sqrt{\Delta}}{4}, \quad \Delta \equiv 0 \pmod{4}
\]

\[
\frac{1}{2} - \frac{\Delta - 1}{4\sqrt{\Delta}}, \quad \Delta \equiv 1 \pmod{4}
\]

It exists only when $\mathcal{O}_K$ is non-Euclidean.
Schmidt Arrangement of \( \mathbb{Q}(\sqrt{-15}) \) with Ghost Circles
Definition
\( \mathcal{P} \subset S'_K, \ C, \ C_1, \ C_2 \in S'_K. \)

1. \( \mathcal{P} \) straddles \( C \) if it intersects the interior and exterior of \( C \).
2. \( C_1 \) and \( C_2 \) are \textit{immediately tangent} if they are externally tangent such that their union straddles no circle of \( S'_K \).
Theorem (S.)

The following are equivalent:

1. \( \mathcal{P} \) is a minimal non-empty set of circles that is closed under immediate tangency.

2. \( \mathcal{P} \) is a maximal tangency-connected set of circles with disjoint interiors and straddling no circle of \( S_K' \).
$K$-Apollonian Packings

$\frac{1 + \sqrt{7}i}{2}$

$\frac{1 + \sqrt{11}i}{2}$
K-Apollonian Packings

Theorem (S.)

*The Schmidt arrangement is the disjoint union of all K-Apollonian circle packings (where circles are oriented).*
The exceptional isomorphism

\[ \rho : \text{PGL}_2(\mathbb{C}) \to \text{SO}_{1,3}^+(\mathbb{R}). \]

- \( \text{SO}_{1,3}^+(\mathbb{R}) \) acts on the 4D real vector space of Hermitian matrices,

\[
\begin{pmatrix}
  b' & x + iy \\
  x - iy & b 
\end{pmatrix}
\]

preserving the determinant, which is a form of signature 3, 1.

- \( \text{PGL}_2(\mathbb{C}) \) acts by conjugation \( \gamma \cdot M = \gamma^\dagger M \gamma \).

- Hermitian forms of determinant 1 (say) ‘are’ circles (take the zero set in \( \hat{\mathbb{C}} \)). This is a hyperboloid in Minkowski space, a model of \( \mathbb{H}^3 \).
The Apollonian Group ($\mathbb{Z}[i]$)

Idea: act on *Descartes quadruples* instead of circles, coded as a $4 \times 4$ matrix

$$W_D = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ \end{pmatrix}$$

**Theorem (GLMWY)**

$C_1, C_2, C_3, C_4$ form a Descartes configuration if and only if

$$W_D^\dagger G_M W_D = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ \end{pmatrix}$$

Codify swaps of Descartes quadruples as a matrix action:

$$W_D \mapsto W_D S_i, \quad i = 1, 2, 3, 4$$

The *Apollonian group* is $\langle S_1, S_2, S_3, S_4 \rangle \subset O_{3,1}(\mathbb{R})$. 
Cheat Sheet for $\mathbb{Q}(i)$

Apollonian group:

$$
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 \\
2 & 0 & 1 & 0 \\
2 & 0 & 0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 2 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 2 & 1 & 0 \\
0 & 2 & 0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 2 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & -1
\end{pmatrix}.
$$

$$\langle r, s, t, u : r^2 = s^2 = t^2 = u^2 = 1 \rangle$$
The Apollonian Group

Codify swaps of Descartes quadruples as a matrix action:

\[ W_D \mapsto W_D S_i, \quad i = 1, 2, 3, 4 \]

The Apollonian group is \( \langle S_1, S_2, S_3, S_4 \rangle \).

1. Freely generated by these four generators of order two.
2. Thin, i.e. infinite index in its Zariski closure \( O_{3,1}(\mathbb{R}) \).
3. Acts freely and transitively on the quadruples in a packing (so packing is orbit of 4 circles).
4. Limit set:
5. Main tool in results on curvatures.
Theorem (S.)

For each imaginary quadratic $K \neq \mathbb{Q}(\sqrt{-3})$, there is a Kleinian group $A < \text{Möb}$ such that

1. Its limit set is the $K$-Apollonian strip packing.
2. It acts freely and transitively on the clusters of any $K$-Apollonian packing (suitably defined).
3. It is finitely generated (with a simple presentation).
4. It is thin.
Cheat Sheet for $\mathbb{Q}(\sqrt{-2})$

Apollonian group:

$$\left\langle r, s, t, u, v, w : r^2 = s^2 = t^2 = u^2 = v^2 = w^2 = 1 \right\rangle$$

$$W_D : v_1, v_3, v_6, v_8$$

$$W_D^\dagger G_M W_D = \begin{pmatrix} 1 & -3 & -3 & -3 \\ -3 & 1 & -3 & -3 \\ -3 & -3 & 1 & -3 \\ -3 & -3 & -3 & 1 \end{pmatrix}$$
Cheat Sheet for $\mathbb{Q}(\sqrt{-7})$

Apollonian group:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 0 & 0 & -1 \\ 3 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 3 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 1 & 2 & 3 & 0 \end{pmatrix}. $$

$$\langle r, s, t : r^2 = s^2 = t^2 = 1 \rangle$$
Cheat Sheet for $\mathbb{Q}(\sqrt{-11})$

Apollonian group:

$$\begin{pmatrix} 1 & 3 & 3 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 3 & 1 & 3 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 3 & 3 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 3 & 3 & 3 & 1 \end{pmatrix}. $$

$$\langle r, s, t, u : r^2 = s^2 = t^2 = u^2 = 1 \rangle$$
Cheat Sheet for $\mathbb{Q}(\sqrt{\Delta})$, $\Delta < -11$

Apollonian group ($\Delta \equiv 0 \pmod{4}$):

$$
\begin{pmatrix}
1 & 2 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 2 & 0 & 1 \\
0 & 2 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 2 \\
0 & 0 & 1 & 2 \\
0 & 1 & 0 & 2 \\
0 & 0 & 0 & -1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 2 & 0 \\
0 & 0 & 2 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 2 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 + \frac{\Delta}{4} & 1 & 1 + \frac{\Delta}{4} \\
0 & 1 & 0 & 0 \\
1 & -\frac{\Delta}{4} & -1 & 0 \\
0 & 0 & 0 & -\frac{\Delta}{4} - 1
\end{pmatrix}.
$$

Apollonian group ($\Delta \equiv 1 \pmod{4}$):

$$
\begin{pmatrix}
1 & -1 & 1 & 1 \\
0 & -1 & 2 & 2 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 1 & -1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 2 & -1 & 2 \\
0 & 1 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 1 & 1 & -1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 2 & 2 & -1
\end{pmatrix},
\begin{pmatrix}
0 & 1 & \frac{\Delta + 3}{4} & 0 \\
1 & 1 & \frac{\Delta - 1}{4} & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & -\frac{\Delta + 3}{4} & 0
\end{pmatrix}.
$$

- Swaps: swap out $v_i$ for $i = 2, 3, 4$ or move $v_1$ to $v_2$.

$\langle r, s, t, u : r^2 = s^2 = t^2 = u^2 = 1, rstu = stur \rangle$
Generalized Local-Global Conjecture

Conjecture (S.)

\( \mathcal{P} \) a primitive, integral \( K \)-ACP for \( K \neq \mathbb{Q}(\sqrt{-3}) \) with discriminant \( \Delta \). Let \( S_M \) be the set of residues of curvatures modulo \( M \). Then, for some \( M \mid 24 \), any sufficiently large integer with a residue in \( S_M \) occurs as a curvature.

A sufficient \( M \) is given by

\[
\begin{align*}
v_2(M) &= \begin{cases} 
3 & \Delta \equiv 28 \pmod{32} \\
2 & \Delta \equiv 8, 12, 20, 24 \pmod{32} \\
1 & \Delta \equiv 0, 4, 16 \pmod{32} \\
0 & \text{otherwise}
\end{cases}, \\
v_3(M) &= \begin{cases} 
1 & \Delta \equiv 5, 8 \pmod{12} \\
0 & \text{otherwise}
\end{cases}.
\end{align*}
\]
Circle Summer: Congruence Subgroups

Joint w/ Andrew Jensen, Cherry Ng, Evan Oliver, Tyler Schrock