The Apollonian structure of Bianchi groups

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Abstract. We study the orbit of \( \mathbb{R} \) under the Bianchi group \( \text{PSL}_2(\mathcal{O}_K) \), where \( K \) is an imaginary quadratic field. The orbit \( \mathcal{S}_K \), called a Schmidt arrangement, is a geometric realisation, as an intricate circle packing, of the arithmetic of \( K \). We define certain natural subgroups whose orbits generalise Apollonian circle packings, and show that \( \mathcal{S}_K \), considered with orientations, is a disjoint union of all of these \( K \)-Apollonian packings. These packings define a new class of thin groups called \( K \)-Apollonian groups. We make a conjecture on the curvatures of these packings, generalizing the local-to-global conjecture of Apollonian circle packings.

1. Introduction

Let \( K \) be an imaginary quadratic field with ring of integers \( \mathcal{O}_K \). The Bianchi group \( \text{PSL}_2(\mathcal{O}_K) \) acts on the extended complex plane \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) via Möbius transformations. Its action permutes the collection of circles of \( \hat{\mathbb{C}} \) (where straight lines are considered circles through \( \infty \)). The orbit of the extended real line \( \hat{\mathbb{R}} \) under \( \text{PSL}_2(\mathcal{O}_K) \) forms a delicately intertwined collection of circles, some examples of which are shown in Figures 1, 4 and 7. This is the Schmidt arrangement of \( K \), denoted \( \mathcal{S}_K \), named in honour of the work of Asmus Schmidt generalizing continued fractions to the complex setting \([24, 25, 26, 27, 28]\). The individual images of \( \hat{\mathbb{R}} \) are called \( K \)-Bianchi circles.

The Schmidt arrangement naturally exhibits various aspects of the number theory of \( \mathcal{O}_K \), and this was the topic of study of an earlier paper of the author \([31]\). For example, \( \mathcal{S}_K \) is connected if and only if \( \mathcal{O}_K \) is Euclidean.

The author’s interest arose from \( \mathcal{S}_{\mathbb{Q}(i)} \) (Figure 1), which made its appearance in \([13]\) as an Apollonian superpacking in the study of Apollonian circle packings (apparently the authors of \([13]\) were unaware of the relation to Schmidt’s work). It is the purpose of this paper to give a new description of this relationship, and introduce some new circle packings arising from the Schmidt arrangements of other imaginary quadratic fields in the same way.

A Descartes quadruple of circles in \( \hat{\mathbb{C}} \) is a quadruple such that every pair of circles is tangent. The curvatures (inverse radii) \( a, b, c, d \) of such a quadruple satisfy the Descartes quadratic relation
\[
(a + b + c + d)^2 = 2(a^2 + b^2 + c^2 + d^2). \tag{1}
\]
The study of Descartes quadruples has a lively history and there are several excellent expositions; see for example \([9, 23]\). If one begins with three mutually tangent circles with curvatures \( a, b, c \), then there are exactly two circles which complete the triple to a Descartes quadruple, with curvatures \( d \) and \( d' \) solving (1); these satisfy
\[
d + d' = 2(a + b + c).
\]
This is referred to as the Descartes rule for generating new circles and curvatures from old. Adding these two new circles to our original triple, we have a set of 5 circles. For each mutually tangent triple in the set, we can again find two circles which complete it to a quadruple. Any...
Figure 1. The Schmidt arrangement of $\mathcal{S}_Q(i)$. The image includes those circles of curvature $\leq 20$ intersecting the closure of the fundamental parallelogram of the ring of integers.

Figure 2. The iteration process generating an Apollonian circle packing.

of these which are not already included in our collection are now added, thereby expanding the set of circles. If we continue this process ad infinitum, we obtain an infinite collection of circles which is called an Apollonian circle packing. See Figure 2. The remarkable fact for the number theorist is that if the first four circles had integer curvatures, then the generation rule entails that every circle in the packing has an integral curvature. For an example, see Figure 3. The natural question is to determine which integers occur as curvatures.

As it happens, $\mathcal{S}_Q(i)$ is the union of all possible primitive, integral Apollonian circle packings, considered up to suitable symmetries and an appropriate scaling (integral refers to integer curvatures; primitive means they share no common factor). This is a result of [13, Theorem 6.3] (using the definition of the Schmidt arrangement as an Apollonian superpacking), further studied in [30]. In particular, the curvatures of $\mathbb{Q}(i)$-Bianchi circles are all integral.

Apollonian circle packings have generated great interest recently, in large part because of their connection to thin groups. The central conjecture is a local-global principle for the curvatures of a packing.

Conjecture 1.1 (Fuchs and Sanden [10], Graham, Lagarias, Mallows, Wilks and Yan [11]). Let $\mathcal{P}$ be a primitive integral Apollonian circle packing, and let $S$ be the set of residue classes modulo 24 of the curvatures in $\mathcal{P}$. Then all sufficiently large integers with residues in $S$ occur as curvatures in $\mathcal{P}$. 
Significant progress has been made toward this conjecture, most notably that it holds for a set of integers of density one [3] (positive density was first shown in [2]). For an excellent overview and further references, see [9]; see also the series of papers [11, 12, 13] which are central to the field, and the exposition [23].

These results depend on an analysis of the Apollonian group, a matrix group which describes the relations between curvatures of tangent circles. The Apollonian group is of infinite index in its own Zariski closure, in other words, it is a thin group. Thin groups are not as accessible as arithmetic groups, but, remarkably, still share some of their properties, most notably a version of strong approximation. The Apollonian group has garnered so much interest in part because of its position as a ‘naturally occurring’ thin group of arithmetic interest. For an overview of the arithmetic of thin groups, and the discoveries rapidly unfolding in recent years, see [17]. It is one of the principal goals of this paper to place the Apollonian group in a new ‘naturally occurring’ infinite family of thin groups of arithmetic interest.

We begin by defining $K$-Apollonian packings for imaginary quadratic fields $K \neq \mathbb{Q}(-3)$. We consider oriented circles, i.e. those which have been assigned an interior and exterior.
Figure 4. Schmidt arrangements $S_K$ of imaginary quadratic fields $K$. Clockwise from top left: $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-7})$, $\mathbb{Q}(\sqrt{-15})$, $\mathbb{Q}(\sqrt{-11})$. In each case, the image includes those circles of curvature $\leq 20$ intersecting the closure of the fundamental parallelogram of $O_K$.

**Definition 1.2.** One says that a collection of circles $\mathcal{P}$ *straddles* a circle $C$ if it intersects both the interior and exterior of $C$ nontrivially. We say that a collection of circles $\mathcal{P}$ is *tangency-connected* if the graph whose vertices are circles and whose edges indicate tangencies is a connected graph. We define a *$K$-Apollonian packing* to be any maximal tangency-connected subset $\mathcal{P}$ of circles in $S_K$ under the condition that $\mathcal{P}$ does not straddle any circle of $S_K$.

For an example, see Figure 5.

Note that, in particular, this implies that the circles in $\mathcal{P}$, oriented appropriately, have disjoint interiors. That is, each circle in $\mathcal{P}$ can be assigned the orientation which excludes $\mathcal{P}$ from its interior.

Now we can state the fundamental relationship between Schmidt arrangements and $K$-Apollonian packings.
Theorem 1.3. For imaginary quadratic $K \neq \mathbb{Q}(\sqrt{-3})$, the Schmidt arrangement $S_K$ is equal to the union of all $K$-Apollonian packings.

The Eisenstein case ($K = \mathbb{Q}(\sqrt{-3})$) presents the difficulty that $S_K$ has circles intersecting other than tangently, as in Figure 7. For this reason, throughout the paper, we will assume that $K \neq \mathbb{Q}(\sqrt{-3})$. This special case is worthy of further investigation.

Let $\Delta < 0$ be the discriminant of $K$. The curvature of a $K$-Bianchi circle is of the form $n\sqrt{-\Delta}$, for some $n \in \mathbb{Z}$; we may call $n$ the reduced curvature of the circle. With this definition, we may extend Conjecture 1.1 for Apollonian circle packings.

Conjecture 1.4. Let $K$ be an imaginary quadratic field with $\Delta \neq -3$. Let $M$ be an integer. Let $\mathcal{P}$ be a $K$-Apollonian packing, and let $S_M$ be the set of residue classes modulo $M$ of the reduced curvatures in $\mathcal{P}$. There exists an $M$ dividing 24 such that all sufficiently large integers whose residues modulo $M$ lie in $S_M$ occur as curvatures in $\mathcal{P}$.

Furthermore, a sufficient $M$ is given by the following formulae for its valuations with respect to 2 and 3:

$$
v_2(M) = \begin{cases} 
3 & \Delta \equiv 28 \pmod{32} \\
2 & \Delta \equiv 8, 12, 20, 24 \pmod{32} \\
1 & \Delta \equiv 0, 4, 16 \pmod{32} \\
0 & \text{otherwise}
\end{cases}, \quad v_3(M) = \begin{cases} 
1 & \Delta \equiv 5, 8 \pmod{12} \\
0 & \text{otherwise}
\end{cases}
$$

Further detailed predictions and supporting evidence is given in Section 16. See Figure 6 for an example packing with curvatures shown.

The bulk of the paper is devoted to the study of $K$-Apollonian packings. In particular, we define $K$-Apollonian groups for every $K \neq \mathbb{Q}(\sqrt{-3})$. Let Möb denote the group of Möbius transformations. Then a $K$-Apollonian group is one satisfying the conclusions of the following theorem.

Theorem 1.5 (Summary of results of Sections 9–15). Let $K \neq \mathbb{Q}(\sqrt{-3})$ be an imaginary quadratic field. Then there exists a Kleinian group $\mathcal{A} < \text{Möb}$ with the following properties:

(1) The limit set of $\mathcal{A}$ is the $K$-Apollonian packing containing $\hat{\mathbb{R}}$. 

Figure 6. An example $\mathbb{Q}(\sqrt{-3})$-Apollonian packing, shown with reduced curvatures (the outer circle has reduced curvature $-1$). In this case, it is conjectured that all sufficiently large integers that are not congruent to 1 modulo 4 appear as reduced curvatures.

(2) Any $K$-Apollonian packing is the orbit under $A$ of some finite collection of circles.

(3) $A$ is of infinite index in its Zariski closure (which is either $SO_{3,1}(\mathbb{R})$ or $O_{3,1}(\mathbb{R})$, under the isomorphism $\text{M"{o}b} \cong O_{3,1}(\mathbb{R})$). In other words, it is thin.

(4) It is possible to define clusters of circles, being unordered collections of $4 \leq n < \infty$ circles in a certain geometric arrangement, so that the set of clusters in any $K$-Apollonian packing is a principal homogeneous space\(^1\) for $A$.

The last property deserves some further explanation. In the study of traditional Apollonian circle packings, the ‘clusters’ are Descartes quadruples (any four circles which are mutually tangent). We have seen that, given three mutually tangent circles, there are exactly two Descartes quadruples containing these three. Therefore, given one Descartes quadruple and one circle of that quadruple, there is a *swap* that replaces that circle with the unique choice that gives a new Descartes quadruple. For each quadruple, there are four such swaps. It

\(^{1}\)By a *principal homogeneous space for a group G* we mean a non-empty set on which the group G acts freely and transitively.
turns out that the space of Descartes quadruples is a principal homogeneous space for the
traditional Apollonian group, whose four generators encode the four swaps. This group has
two manifestations, sometimes called algebraic and geometric. The geometric manifestation is
the one discussed in the theorem above, but both aspects are discussed in the paper.

There are many \( K \)-Apollonian groups (even for traditional Apollonian circle packings), and
we give some particularly simple and symmetric examples with pleasing presentations. In
particular, when \( O_K \) is Euclidean, we give an example which is a free product of finitely many
copies of \( \mathbb{Z}/2\mathbb{Z} \). In each case, we give generators explicitly, and describe their interpretation as
‘sparse’ or ‘clusters’.

The methods of the paper give rise to a few results that may be of independent interest.
The following theorem strengthens the results of [31].

**Theorem 1.6** (Section 5). Suppose \( O_K \) is non-Euclidean. Then the tangency graph of \( S_K \) is
an infinite forest of trees of infinite valence.

The following strengthens results of [21].

**Theorem 1.7** (Theorem 9.5). Whenever \( O_K \) is non-Euclidean, the subgroup of \( \text{PSL}_2(O_K) \)
generated by elementary matrices is a thin group.

And finally, the paper contains in Section 10 a discussion of topographical groups. These are
subgroups of \( \text{PGL}_2(\mathbb{Z}) \) for which unordered superbases are a principal homogeneous space.
There are only two such groups, which are isomorphic under the outer automorphism of
\( \text{PGL}_2(\mathbb{Z}) \). They have a Cayley graph isomorphic to the topograph of Conway and Fung [4].

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**A note on the figures.** The figures in this paper were produced with Sage Mathematics
Software [32]. Figures 2, 4, 7 and 8 appeared previously in [30, 31].
2. Notations

Throughout the paper, $K$ is an imaginary quadratic field with discriminant $-3 \neq \Delta < 0$ and ring of integers $\mathcal{O}_K$. The ring $\mathcal{O}_K$ has an integral basis $1, \tau$, where
\[
\tau^2 = \begin{cases} \Delta/4 & \Delta \equiv 0 \pmod{4} \\ \tau + (\Delta - 1)/4 & \Delta \equiv 1 \pmod{4} \end{cases}.
\]
It is convenient to write
\[
\epsilon = \text{Tr}(\tau) = \begin{cases} 0 & \Delta \equiv 0 \pmod{4} \\ 1 & \Delta \equiv 1 \pmod{4} \end{cases}.
\]
In other words,
\[
\tau, \bar{\tau} = \frac{\epsilon \pm \sqrt{\Delta}}{2}.
\]
Write $N : K \to \mathbb{Q}$ for the norm map $\alpha \mapsto \alpha \bar{\alpha}$. In particular,
\[
N(\tau) = -\frac{\Delta - \epsilon}{4}.
\]

We follow [12] in writing M"ob for the conformal group, the group of conformal maps of $\hat{\mathbb{C}}$, including Möbius transformations and reflections (i.e. allowing complex conjugation); and writing $\text{M"ob}_+$ for the group of Möbius transformations without reflections, which is isomorphic to $\text{PSL}_2(\mathbb{C})$ via the usual matrix representation of a Möbius transformation. We also write $\text{GM}^* := \text{M"ob} \times \{ \pm I \}$ for the extended Möbius group. The group M"ob is isomorphic to the isometry group of three-dimensional hyperbolic space, $\mathbb{H}^3$, via its action on the boundary of the upper-half-space model.

**Warning:** For the remainder of the paper, $K \neq \mathbb{Q}(\sqrt{-3})$.

3. Preliminaries

We recall some basic facts about Schmidt arrangements from [31], to which the reader is referred for further details. Let $K$ be an imaginary quadratic field. Recall that throughout the paper, $K \neq \mathbb{Q}(\sqrt{-3})$. Among the remaining cases, $K = \mathbb{Q}(i)$ is special in several regards, as we will describe.

**Definition 3.1.** If $M \in \text{PSL}_2(\mathcal{O}_K)$, then $M(\hat{\mathbb{R}})$ is called a $K$-Bianchi circle. The collection of all $K$-Bianchi circles is called the Schmidt arrangement, denoted $S_K$. We will also refer to the union of these circles as $\hat{S}_K$, without fear of confusion.

It is convenient to consider oriented circles.

**Definition 3.2 ([31, Definitions 3.1, 3.2, 3.3]).** An oriented circle is a circle together with an orientation, which is a direction of travel, specified as either positive/counterclockwise or negative/clockwise. The interior of an oriented circle is the area to your left as you travel along the circle according to its orientation. For lines (circles through $\infty$) besides $\hat{\mathbb{R}}$, positive orientation indicates travel in the direction of increasing imaginary part. For $\hat{\mathbb{R}}$, positive orientation indicates travel to the right. An oriented $K$-Bianchi circle is an oriented circle whose underlying circle is a $K$-Bianchi circle. Write $\hat{S}_K$ for the collection of oriented $K$-Bianchi circles.

The map $\hat{S}_K \to S_K$ which forgets orientation is two-to-one.

Let us also set the convention that $\hat{\mathbb{R}}$, when considered an oriented circle, denotes the positively oriented circle (whose interior is the upper half plane). The group $\text{PGL}_2(\mathbb{C})$ has a natural action on oriented circles via Möbius transformation, so that, with this convention, $M(\hat{\mathbb{R}})$ is naturally endowed with an orientation. The stabilizer of $\hat{\mathbb{R}}$ is $\text{PSL}_2(\mathbb{R})$. Furthermore, by Proposition 3.4 of [31], this action restricts, in the case $K \neq \mathbb{Q}(i)$, to a transitive action of $\text{PGL}_2(\mathcal{O}_K)$ on oriented $K$-Bianchi circles, with $\text{Stab}(\hat{\mathbb{R}}) = \text{PSL}_2(\mathbb{Z})$. In the case of $K = \mathbb{Q}(i)$, the same holds with $\text{PSL}_2(\mathcal{O}_K)$ in place of $\text{PGL}_2(\mathcal{O}_K)$ (the stabilizer remains the same; the essential
fact here is that for this case, \( \text{PSL}_2(O_K) \) already contains both orientations of each \( K \)-Bianchi circle. If one considers the orbit of \( \hat{\mathbb{R}} \) under \( \text{PGL}_2(\mathbb{Z}[i]) \), one obtains a strictly larger collection of circles than the Schmidt arrangement. However, the new circles are not interesting: the set consists of two copies of the Schmidt arrangement at right angles. See [31, Section 3].

Oriented circles are best described as parameterized by four real parameters, as follows.

**Proposition 3.3** ([31, Proposition 3.5]). Let \( C \) be an oriented circle in \( \hat{\mathbb{C}} \) (including those through \( \infty \)). Then the circle \( C \) can be given uniquely in the form

\[
\left\{ \frac{X}{Y} \in \hat{\mathbb{C}} : bX \overline{X} - aY \overline{X} - \pi X \overline{Y} + b'Y \overline{Y} = 0 \right\}.
\]

where the following hold:

1. \( b, b' \in \mathbb{R}, a \in \mathbb{C} \),
2. we have \( b'b = a\overline{a} - 1 \),
3. \( b \) has sign equal to the orientation of \( C \), and;
4. if \( b = b' = 0 \), in which case \( C \) must be a line, then \( a \), as a vector, is a unit vector pointing from exterior to interior orthogonal to \( C \).

Furthermore, if \( C' \) is the image of \( C \) under \( z \mapsto 1/z \), and \( C \) has parameters \((b, b', a)\) according to the requirements above, then \( C' \) has parameters \((b', b, \overline{a})\) according to the requirements above.

Finally,

1. \( b = 0 \) if and only if \( \infty \in C \),
2. \( b' = 0 \) if and only if \( 0 \in C \),
3. if \( \infty \notin C \), then \( C \) has radius \( 1/|b| \),
4. if \( \infty \notin C \), then \( C \) has centre \( a/b \).

**Definition 3.4** ([31, Definition 3.6]). For any circle \( C \) in \( \hat{\mathbb{C}} \), expressing it as in Proposition 3.3, we call \( b \) the curvature (elsewhere sometimes called a bend), \( b' \) the co-curvature, and \( a \) the curvature-centre.

**Proposition 3.5** ([31, Proposition 3.7]). Consider an oriented circle expressed as the image of \( \hat{\mathbb{R}} \) under a transformation of the form

\[
M = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}, \quad |\alpha \delta - \beta \gamma| = 1.
\]

The curvature of the circle is given by

\[
i(\beta \overline{\delta} - \beta \delta),
\]

the co-curvature of the circle is given by

\[
i(\alpha \overline{\gamma} - \alpha \gamma),
\]

and the curvature-centre is given by

\[
i(\alpha \overline{\delta} - \gamma \overline{\beta}).
\]

The curvature and co-curvature of an oriented \( K \)-Bianchi circle are integer multiples of \( \sqrt{-\Delta} \); the integer alone will be referred to as the reduced curvature or reduced co-curvature, respectively.

**Proposition 3.6** ([31, Propositions 4.1 and 4.4]). Two \( K \)-Bianchi circles may intersect only at \( K \)-points, and only tangently\(^2\).

It is useful to be very explicit about which circles are tangent. We say that \( \alpha, \beta \in O_K \) are coprime if the ideals they generate are coprime, i.e. \((\alpha) + (\beta) = (1)\).

\(^2\)This does not hold for \( K = \mathbb{Q}(\sqrt{-3}) \). The reader is reminded, one final time, that \( K \neq \mathbb{Q}(\sqrt{-3}) \) throughout the paper.
Proposition 3.7 ([31, Proposition 4.3]). Let $\alpha/\beta \in K$ be such that $\alpha$ and $\beta$ are coprime. Suppose $|O_K^*| = n$. Then the collection of oriented $K$-Bianchi circles passing through $\alpha/\beta$ is a union of $n$ generically different $\mathbb{Z}$-families, one for each $u \in O_K$. The family associated to $u$ consists of the images of $\mathbb{R}$ under the transformations
\[
\begin{pmatrix} \alpha & w\gamma + k\tau\alpha \\ \beta & u\delta + k\tau\beta \end{pmatrix}, \quad k \in \mathbb{Z},
\]
where $\gamma, \delta$ is a particular solution to $\alpha\delta - \beta\gamma = 1$. Furthermore,

1. The curvatures of the circles in one family form an equivalence class modulo $\sqrt{-\Delta N(\beta)}$.
2. The centres of the circles in a given family lie on a single line through $\alpha/\beta$.
3. The family given by unit $u$ contains the same circles as the family given by $-u$, but with opposite orientations.

4. $K$-Apollonian packings and immediate tangency

In this section we give two equivalent definitions of a $K$-Apollonian packing. This allows us to prove Theorem 1.3 of the introduction, stating that a Schmidt arrangement is the disjoint union of its $K$-Apollonian packings.

Definition 4.1. Let $P$ be a subset of $S_K$ or $\hat{S}_K$. The tangency graph of $P$ is the graph whose vertices are the circles (oriented if appropriate) of $P$ and whose edges indicate tangencies. We say that $P$ is tangency-connected if its tangency graph is connected. We say $P$ straddles a circle $C$ if it intersects both the interior and exterior of $C$ nontrivially.

This definition of straddling coincides with that of [31, Definition 4.5] in the cases we are considering (where circles intersect only tangently), and is simpler to state.

We repeat Definition 1.2 of the introduction here:

Definition 4.2. We define a $K$-Apollonian packing of unoriented circles to be any maximal tangency-connected subset $P$ of circles of $S_K$ under the condition that $P$ does not straddle any circle of $S_K$.

We will now extend the definition of a $K$-Apollonian packing to oriented circles.

Definition 4.3. We define a $K$-Apollonian packing of oriented circles to be any maximal tangency-connected subset $P$ of oriented circles of $\hat{S}_K$ with disjoint interiors under the condition that $P$ does not straddle any circle of $\hat{S}_K$.

The following lemma verifies that these two definitions correspond nicely under the orientation-forgetting map.

Lemma 4.4. A collection of circles $P \subset S_K$ is a $K$-Apollonian packing of unoriented circles if and only if it can be obtained from a $K$-Apollonian packing of oriented circles by forgetting orientation.

Proof. Suppose $P \subset S_K$ does not straddle any circle of $S_K$. Then, in particular, it cannot intersect both the interior and exterior of any $C \in P$. Therefore, $P$ can be lifted to $P' \subset \hat{S}_K$ by choosing the orientation of each circle $C \in P$ in such a way that $P$ is disjoint from the interior of $C$. This lift is unique if $P$ contains at least two circles. The resulting collection has disjoint interiors and the non-straddling property. Furthermore, $P'$ is tangency-connected if and only if $P$ is.

Therefore, it remains to show that $P$ is non-straddling and maximal with respect to being tangency-connected if and only if $P'$ is non-straddling, maximal with respect to being tangency-connected and has disjoint interiors.

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3 The circles in question needn’t have orientations to make this definition.
A $K$-Apollonian packing in $\hat{S}_K$ cannot contain both orientations of the same unoriented circle without violating the disjoint interiors requirement (unless it contained only those two circles, but then it would not be maximal). Therefore, if $P$ is a $K$-Apollonian packing, then $P'$ is a $K$-Apollonian packing. For, if there is a way to add another circle in $\hat{S}_K$ it corresponds to a way to add another circle in $S_K$.

Conversely, if $P'$ is a $K$-Apollonian packing, then $P$ is a $K$-Apollonian packing, since if we could add another circle to $P$ it could be lifted to $P'$.

With this result in place, we may henceforth consider oriented circles. We wish to give a different characterisation of being a $K$-Apollonian packing of oriented circles, and show that it is equivalent. We will need the following notion.

**Definition 4.5.** Two oriented $K$-Bianchi circles $C_1, C_2 \in \hat{S}_K$ are immediately tangent if they are externally tangent in such a way that the pair straddles no circles of $\hat{S}_K$.

We recall the following result, which shows that for any $K$-Bianchi circle, there is exactly one $K$-Bianchi circle immediately tangent to it at a fixed $K$-rational point.

**Proposition 4.6 ([31, Proposition 6.2]).** Let $C \in \hat{S}_K$ be an oriented $K$-Bianchi circle with $K$-rational point $x$. Then there exists

$$M_C = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \in \text{PGL}_2(\mathcal{O}_K)$$

such that $C = M_C(\mathbb{R})$ and $x = \alpha/\beta$. Furthermore, there exists exactly one oriented $K$-Bianchi circle $C' \in \hat{S}_K$ immediately tangent to $C$ at $x$, given by $C' = M_C'(\mathbb{R})$ where

$$M_{C'} = \begin{pmatrix} \alpha & -\gamma + \tau\alpha \\ \beta & -\delta + \tau\beta \end{pmatrix} = M_C \begin{pmatrix} 1 & \tau \\ 0 & -1 \end{pmatrix} \in \text{PGL}_2(\mathcal{O}_K).$$

**Definition 4.7.** Let $P$ be a subset of $\hat{S}_K$. We say that $P$ is closed under immediate tangency if it has the property that for any circle $C \in P$, any circle $C' \in \hat{S}_K$ that is immediately tangent to $C$ is also contained in $P$. We say that $P$ is a $K$-Apollonian packing defined by tangency if it is minimal among non-empty subsets of $\hat{S}_K$ which are closed under immediate tangency.

There are infinitely many circles immediately tangent to any one circle, by Proposition 4.6, so $K$-Apollonian packings defined by tangency are infinite sets of circles.

This definition can be rephrased slightly, in the language of graph theory.

**Definition 4.8.** The immediate tangency graph of $\hat{S}_K$ is the graph whose vertices are the circles of $\hat{S}_K$ and whose edges are given by the symmetric relation of immediate tangency.

The immediate tangency graph is naturally a subgraph of the tangency graph, having the same vertices.

**Proposition 4.9.** A subset $P$ of $\hat{S}_K$ is a $K$-Apollonian packing defined by tangency if and only if it is a connected component of the immediate tangency graph.

The proof is immediate.

**Proposition 4.10.** A subset $P$ of $\hat{S}_K$ is a $K$-Apollonian packing if and only if it is a $K$-Apollonian packing defined by tangency.

**Proof.** Let $P$ be a $K$-Apollonian packing. Let $C$ be a circle of $P$ and let $z \in C$ be a $K$-rational point. Let $C'$ be the unique circle immediately tangent to $C$ at $z$. Our first goal is to show that $P$ contains $C'$. This demonstrates that $P$ is closed under immediate tangency.

If not, since $C$ is in the exterior of $C'$, then all of $P$ is exterior to $C'$. Suppose we add $C'$ to $P$. This creates a larger tangency-connected set with disjoint interiors. We show that this
interiors. Suppose not: then some circle \( C \) connected, we can assume without loss of generality that

We have shown that \( P \) is not minimal; its circles inside \( D \) could be removed without violating closure under immediate tangency. This is a contradiction. Therefore we have shown that \( P \) straddles no circles.

We have already observed that \( P \) is tangency-connected. Next we show that \( P \) has disjoint interiors. Suppose not: then some circle \( C \) lies inside another, \( C' \). Since \( P \) is tangency-connected, we can assume without loss of generality that \( C \) is tangent to \( C' \) at some point \( z \). But since \( P \) also includes the circle immediately tangent to \( C' \) at \( z \), this gives a collection of three distinct circles tangent at a point. This is disallowed by the non-straddling property just shown.

Finally, we must show \( P \) is maximal. Suppose there were another circle \( C \notin P \) such that \( P \cup \{C\} \) is tangency-connected. Then we will show \( P \cup \{C\} \) straddles some circle of \( S_K \). Let \( z \in C \) be a point of tangency with a circle \( D \in P \). This is not an immediate tangency, since \( C \notin P \). Then there is some \( C' \neq D, C' \in P \), immediately tangent to \( C \). Thus we have three distinct circles of \( P \cup \{C\} \) sharing a single tangency point: as a set they must straddle one of their members.

The following statement is an immediate consequence of the two equivalent definitions.

**Theorem 4.11.** The \( K \)-Apollonian packings of \( \hat{S}_K \) form a collection of disjoint subsets of \( \hat{S}_K \) whose union is all of \( \hat{S}_K \).

**Proof.** This is evident from Definition 4.7, as the \( K \)-Apollonian packings are exactly the connected components of the graph on \( \hat{S}_K \) given by the symmetric relation of immediate tangency.

**Proof of Theorem 1.3.** Forget orientations in Theorem 4.11.

Note that the packings are not disjoint as collections of unoriented circles, however. A circle generally belongs to two packings, depending on the orientation assigned: one living in its interior and another in its exterior.

It is an open question whether tangency-connectedness corresponds exactly to the topological notions of connectedness or path-connectedness for subsets of Schmidt arrangements (it is clearly stronger for arbitrary unions of circles in \( \hat{C} \)). See [31, Section 7] for more on this distinction. In particular, it is shown in [31, Theorem 7.1] that \( S_K \) is connected if and only if it is tangency-connected if and only if \( O_K \) is Euclidean.

**Definition 4.12.** The \( K \)-Apollonian packing containing \( \hat{R} \) is called the fundamental packing, and is denoted \( P_K \).
See Figure 5 for the fundamental packing of $\mathbb{Q}(i)$. Other fundamental packings are shown in Figures 16, 19 and 22.

5. LOOPS IN THE TANGENCY GRAPH

The purpose of this section is to prove the following theorem, from which Theorem 1.6 of the introduction follows (note that [31, Theorem 7.5] already guarantees that $S_K$ has infinitely many components when $\Delta \leq -15$).

**Theorem 5.1.** Let $K$ be such that $\Delta \leq -15$. Then the tangency graph of $\hat{S}_K$ contains no loops (hence neither does the immediate tangency graph).

The proof for $|\Delta| \geq 16$ will rely on a simple graph theory principle.

**Lemma 5.2.** Consider a graph $G$ with vertex set $V$ and a function $f : V \rightarrow \mathbb{R}$. Direct each edge of $G$ according to the direction of increase of $f$ (whenever it is not constant). Suppose that at any vertex $v \in V$, all but at most one edge is directed outward. Then $G$ contains no loops.

Of course, the same statement holds if 'outward' is replaced with 'inward'.

**Proof.** Suppose there is a loop. Let $v$ be a vertex of that loop. By assumption, at least one of the two edges of the loop adjacent to $v$ is directed outward. Moving to the next vertex $w$ along this edge, we enter $w$ along an inward directed edge, and therefore, continuing along the loop, leave $w$ along an outward directed edge. Hence, as we travel around the loop in this direction, the value of $f$ is increasing, so we can never return to $v$. □

**Lemma 5.3.** Let $K$ be such that $|\Delta| > 16$. Let $C$ be a $K$-Bianchi circle. Then of the $K$-Bianchi circles tangent to $C$, there is at most one with curvature less than or equal to the curvature of $C$.

**Proof.** Suppose $C = M(\hat{\mathbb{R}})$ for

$$M = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix},$$

and let $\Lambda = \beta \mathbb{Z} + \delta \mathbb{Z}$. Let $b$ be the reduced curvature of $C$. Then the covolume of $\Lambda$ as a lattice in $\mathbb{C}$ is $|b\sqrt{-\Delta}/2|$. There is at most one element of $\Lambda$ whose norm is less than its covolume, so for all but one $x \in \Lambda$, we have

$$N(x) \geq |b\sqrt{-\Delta}/2|. \quad (3)$$

On the other hand, if $C'$ is tangent to $C$ at $x$ and has curvature $b'$ where $|b'| \leq |b|$, then

$$|kN(x) - b| = |b'| \leq |b|$$

for some $k \in \mathbb{Z}^{>0}$, whence

$$N(x) \leq 2|b|. \quad (4)$$

Comparing the inequalities (3) and (4) gives the desired result. □

The proof for $K = \mathbb{Q}(\sqrt{-15})$ requires an entirely different method of proof.

**Proposition 5.4.** The tangency graph of $K = \mathbb{Q}(\sqrt{-15})$ contains no loops.

**Proof.** Let $K = \mathbb{Q}(\sqrt{-15})$. First, we remark that $\text{PSL}_2(\mathcal{O}_K)$ is transitive on pairs $(C, z)$ where $C \in S_K$ and $z$ is a tangency point of $C$ within $S_K$. Thus, we need only show that $\hat{\mathbb{R}}$ (with interior below) and any $C \in S_K$ of the form $ki + \hat{\mathbb{R}}$ (with interior above) do not participate as adjacent vertices in any cycle of the graph. Suppose for a contradiction, that they do. We will call this the postulated cycle.

Note that $k \geq \sqrt{15}/4$ (equality occurs if $C$ is the circle immediately tangent to $\hat{\mathbb{R}}$ at $\infty$). Then the finitely many other circles making up the cycle must form a chain of tangent circles reaching from some tangency point $x$ of $\hat{\mathbb{R}}$ up to some tangency point $y$ of $C$. In particular, the chain...
Figure 8. The Schmidt arrangement of $Q(\sqrt{-15})$, in blue, with ghost circles, shown in yellow.

consists of finitely many circles. Each is of curvature at least $\sqrt{15}$ or else equal to 0. Therefore, the portion of the chain that lies below $\hat{R} + \frac{\sqrt{-15}}{2}$ is bounded away from $\infty$.

In [31, Section 7], it was shown that $S_K$ is disconnected by demonstrating the existence of a ghost circle which is not tangent to any circle of $S_K$. The ghost circle $G$ is the circle of radius $\frac{1}{\sqrt{15}}$ centred at

$$\frac{1}{2} - \frac{7\sqrt{-15}}{30}.$$ 

Its existence immediately implies the existence of infinitely many other ghost circles formed by $\text{PSL}_2(\mathcal{O}_K)$ images of $G$. The union of these circles is contained in the complement of $S_K$. These images include an infinite ‘ghost chain’ of tangent circles extending horizontally to $\infty$ in both directions, and separating $\hat{R}$ and $\hat{R} + \frac{\sqrt{-15}}{2}$. To see this, illustrated in Figure 8, let $G'$ be the reflection of $G$ in $\hat{R}$, and let $G''$ be the translation of $G$ by $\frac{-1+\sqrt{-15}}{2}$. The circles $G'$ and $G''$ are tangent at $\frac{1+\sqrt{-15}}{4}$. The union of translates of the pair $G'$ and $G''$ by $\mathbb{Z}$ is the ‘ghost chain’ of tangent circles which contradicts the existence of the postulated cycle, and the theorem is proved.

Proof of Theorem 5.1. It suffices to demonstrate this for the tangency graph, as the immediate tangency graph is obtained by removing edges. By Lemma 5.3, we may apply Lemma 5.2 with the tangency graph of $\hat{S}_K$. Direct the edges of $G$ in the direction of increase of curvatures (this may leave some edges undirected). This proves the theorem for $|\Delta| > 16$. The only remaining case of $\Delta = -15$ is completed by Proposition 5.4. □
In the Euclidean cases, we will see that the immediate tangency graph does contain loops.

### 6. The space of circles

We introduce an embedding of the space of circles in Minkowski space which is described in [12] and elaborated upon beautifully by Kocik [16]; it is a natural viewpoint from the perspective of the spin homomorphism and hyperbolic space as the space of Hermitian forms; see, for example, [7]. Associating a circle to a Hermitian form goes back to Bianchi [5, Chapter XV, p. 272]. See also [15].

Let $M$ be the vector space $\mathbb{R}^4$ endowed with an inner product of signature $3,1$ given by the Gram matrix

\[
G_M = \begin{pmatrix}
0 & -\frac{1}{2} & 0 & 0 \\
-\frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

The associated quadratic form will be denoted $M$, for Minkowski. This space is identified with the collection of Hermitian matrices via

\[
\begin{pmatrix} b' \\ b \\ r \\ m \end{pmatrix} \leftrightarrow \begin{pmatrix} b' \\ r - mi \\ r + mi \\ b \end{pmatrix}.
\]

Under this identification, the self-product of a vector under $G_M$ is equal to the determinant of the corresponding matrix. In what follows, we will use this identification implicitly.

The space $\hat{M}$ (considered as Hermitian matrices $T$) comes with a metric-preserving action of $\text{PGL}_2(\mathbb{C})$ via

\[
\gamma \cdot T := \frac{\gamma T \gamma^\dagger}{N(\det \gamma)}, \quad \gamma \in \text{PGL}_2(\mathbb{C}),
\]

where $\dagger$ indicates the conjugate transpose. There is also an action by $\text{Möb}$, where complex conjugation $c$ acts by $c \cdot T = T$. This allows us to define a map

\[
\rho : \text{Möb} \to O^+_M(\mathbb{R})
\]

where $O^+_M(\mathbb{R})$ is the collection of time-preserving isometries of $M$ (in our description, this is equivalent to preserving the sign of $b + b'$), also known as the orthochronous Lorentz group. This map is an isomorphism, which we call the _exceptional isomorphism_, closely related to the famous spin homomorphism. It can also be extended to $G^*$, so that we have the following isomorphisms and subgroup relations:

\[
\hat{\text{Möb}} \rightarrow \hat{\text{Möb}} \rightarrow \hat{\text{GM}} \rightarrow \hat{G}^*
\]

\[
\rho \cong \rho \cong \rho \cong
\]

\[
\hat{\text{SO}}^+_M(\mathbb{R}) \rightarrow \hat{O}^+_M(\mathbb{R}) \rightarrow \hat{O}(\mathbb{R})
\]

For more on this, see [12].

Two Hermitian forms $H_i(u,v)$, $i = 1,2$, on a $\mathbb{C}$-vector space $V$ are usually called _equivalent_ or _isometric_ over $\mathbb{C}$ if $H_1(\phi(u),\phi(v)) = H_2(u,v)$ for some $\mathbb{C}$-isomorphism $\phi : V \to V$. The equivalence class of a Hermitian form and its determinant form a complete set of invariants for the orbits of the $\text{PGL}_2(\mathbb{C})$-action on Hermitian forms described above. All Hermitian forms of positive determinant are equivalent over $\mathbb{C}$ (see, for example, [19] and [7, Chapter 9]). Therefore the action of $\text{PGL}_2(\mathbb{C})$ on the locus $M = 1$ is transitive.

Let $\text{Circ}$ be the collection of circles in $\hat{\mathbb{C}}$. This is a set with a Möb action (where $c$ acts by conjugation on $\hat{\mathbb{C}}$). Denote by $\hat{\text{Circ}}$ the collection of oriented circles, which is also a set with an action by Möb.
Figure 9. The angle of intersection of two circles.

We define a Pedoe map

\[ \pi : \hat{\text{Circ}} \to \mathbb{M} \]

which will endow the space of circles with the structure of an inner product space (see [16]). The map is defined as follows:

\[ \pi(C) = \begin{pmatrix} b' & r + mi \\ r - mi & b \end{pmatrix} \leftrightarrow \begin{pmatrix} b' \\ b' r \\ r \\ m \end{pmatrix}, \]

where \( b \) is the curvature, \( b' \) the co-curvature, and \( r + mi \) the centre-curvature of \( C \). The image of \( \pi \) is a Hermitian matrix of determinant 1 (by Proposition 3.3), hence the image lies on the hyperboloid \( M = 1 \) in \( \mathbb{M} \).

**Proposition 6.1.** The map \( \pi : \hat{\text{Circ}} \to \{ v \in \mathbb{M} : M(v) = 1 \} \) is a M"ob-equivariant bijection via \( \rho \).

**Proof.** Bijectivity follows from Proposition 3.3 (note that \( M(v) = 1 \) is exactly (2)). That \( \pi \) respects the M"ob-action is a direct computation. \( \square \)

In particular, the oriented \( K \)-Bianchi circles are in bijection with the orbit of \( (0, 0, 0, 1)^t \in \mathbb{M} \) under \( \rho(\text{M"ob}(O_K)) \).

One may verify that \( \pi \) may also be computed in the following way. Express a circle \( C \) as \( C = M_C(R) \) for \( M_C = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \in \text{PGL}_2(\mathbb{C}) \), \( N(\det(M_C)) = 1 \).

Then

\[ \pi(C) = i(N - N^\dagger), \text{ where } N = \begin{pmatrix} \alpha \gamma & \alpha \delta \\ \beta \gamma & \beta \delta \end{pmatrix}. \]

It is a brief computation that this map agrees with \( \pi \). Note that, if we write \(-C\) for the circle \( C \) with opposite orientation, then \( \pi(-C) = -\pi(C) \).

The inner product on Minkowski space now gives us an inner product on circles. It carries geometric information about the circles in question. Following Kocik, we will call this the **Pedoe product**.

**Proposition 6.2** (Proposition 2.4 of [16]). Let \( v_i = \pi(C_i) \) for two circles \( C_1, C_2 \) which are not disjoint. Then \( \langle v_1, v_2 \rangle = \cos \theta \), where \( \theta \) is the angle between the two circles as in Figure 9. In particular,

1. \( \langle v_1, v_2 \rangle = -1 \) if and only if the circles are tangent externally

\[ \circ \circ \]

2. \( \langle v_1, v_2 \rangle = 1 \) if and only if the circles are tangent internally

\[ \circ \circ \]
(3) $\langle v_1, v_2 \rangle = 0$ if and only if the circles are mutually orthogonal. This extends via the Law of Cosines, so that the inner product of disjoint circles is

$$\langle v_1, v_2 \rangle = \frac{1}{2} \left( d^2 b_1 b_2 - b_2 / b_1 - b_1 / b_2 \right),$$

where $b_1$ and $b_2$ are the curvatures and $d$ is the distance between centres.

7. The Gaussian integers and Descartes configurations

The study of Apollonian circle packings depends on the study of the action of the so-called Apollonian group on the space of Descartes configurations. We will develop a similar theory for $K$-Apollonian packings in other imaginary quadratic fields. Therefore, we will review briefly the case of Apollonian circle packings for comparison. See [12] for details.

We will consider Apollonian circle packings formed from circles in $\hat{\mathbb{C}}$ (any Apollonian circle packing may be realized this way). Let $W_D$ be a matrix whose columns are $\pi(C_i)$ for any four oriented circles $D : C_1, C_2, C_3, C_4$. By Proposition 6.2, these four circles are in Descartes configuration, or this is the case once all orientations are reversed, if and only if

$$W_D^t G M W_D = R.$$

(6)

Let $D$ denote the space of matrices satisfying this equation, known as the space of Descartes configurations. Since $G_M$, $R$, and the elements of $D$ are all invertible, we obtain

$$W_D R^{-1} W_D^t = G_M^{-1}.$$

This collection of 16 quadratic equations in the curvatures, co-curvatures and centre-curvatures of four circles are the full ‘Descartes relations’ as in [18]: the circles are in Descartes configuration if and only if these 16 equalities hold. In particular, the upper left corner of this matrix equality is the relation on curvatures (1).

Given four circles in Descartes configuration (write $v_1, v_2, v_3, v_4 \in \mathbb{M}$), and a chosen subset of three of them, say $v_1, v_2, v_3$, the fourth circle $v_4$ can be swapped out for its alternative $v'_4$, i.e. the unique other circle which forms a Descartes configuration with $v_1, v_2, v_3$. This takes one configuration in an Apollonian circle packing $\mathcal{P}$ to another in the same packing. The relation that describes this swap in $\mathbb{M}$ is very simple:

$$v_i + v'_i = 2 \left( \sum_{j \neq i} v_j \right).$$

In particular, if the original curvatures are $a, b, c, d$, then the new curvature is

$$d' = 2(a + b + c) - d.$$

There are four ways to swap out a circle. This is accomplished by right multiplication on $D$ by the following four matrices of $O_R(\mathbb{R})$ of order two:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}. $$

(7)

Right multiplication by one of the matrices (7) corresponds to inversion in a circle orthogonal to three of the four circles of the quadruple, as in Figure 10. For example, the four swaps

4Note that these are ordered, oriented configurations in the sense of [13].
on the so-called base cluster shown in Figure 11 are accomplished by the following Möbius transformations (given in the same order as (7)):  

$$
z \mapsto \frac{(2i + 1)\pi - 2}{2\pi + 2i - 1}, \quad z \mapsto -\pi + 2, \quad z \mapsto \frac{-\pi}{2\pi - 1}, \quad z \mapsto -\pi, \quad (8)
$$

The group generated by the matrices (7) is commonly referred to as the algebraic Apollonian group, which we will denote $\hat{A}_{Q(i)}$. The group generated by the Möbius transformations (8) is commonly referred to as the geometric Apollonian group, which we will denote $A_{Q(i)}$. While the algebraic Apollonian group does not act on individual circles, only ordered quadruples, the geometric Apollonian group is a group of Möbius transformations acting on circles, unordered or ordered quadruples. However, both groups have the property that the orbit of the base quadruple gives the full Apollonian packing. Furthermore, the two are isomorphic as groups. The following section will elaborate on the isomorphism.

**Theorem 7.1** (Graham, Lagarias, Mallows, Wilks, Yan [12, Proof of Theorem 4.3]). There are no relations on the matrices of (7) besides the fact that they are of order two. Therefore, the group $\hat{A}_{Q(i)}$, and also $A_{Q(i)}$, is a free product of the four copies of $\mathbb{Z}/2\mathbb{Z}$.

This group is the basic tool in arithmetic results concerning curvatures in Apollonian circle packings, because of the following.

**Theorem 7.2** (Graham, Lagarias, Mallows, Wilks, Yan [12, Theorem 4.3]). Let $\mathcal{P}$ be an Apollonian circle packing.

1. The full set of Descartes configurations contained in $\mathcal{P}$ is a union of 48 orbits of $\hat{A}_{Q(i)}$.
2. Fix a Descartes quadruple $D \in \mathcal{P}$. Then there are 48 matrices $W_D \in \mathcal{D}$ (i.e., satisfying (6)) representing this quadruple. Each of the 48 orbits contains exactly one of these 48 matrices.

The 48 matrices representing a quadruple are formed by reordering the circles 24 ways, and reversing the orientation of all four simultaneously.
8. Cluster spaces and the algebraic-geometric correspondence

Definition 8.1. A cluster space is a set of the form

\[ S_R := \{ W \in M_{4 \times 4}(\mathbb{R}) : W^t G_M W = R \}, \]

where \( R \) is a fixed invertible matrix.

The space \( D \) of Descartes quadruples of the last section is such a space. When the columns of the \( W \) lie on \( M = 1 \), this can be considered the collection of quadruples of circles, considered in \( M \), which are in a particular configuration (specified by \( R \)) with respect to the Pedoe product. In general, we loosen this requirement on the columns, allowing them to represent linear combinations of circles.

The purpose of the next two results is to show that \( S_R \) is a principal homogeneous space under left and right actions on \( S_R \) by matrix groups isomorphic to \( O_M(\mathbb{R}) \).

Definition 8.2. The left action of \( O_M(\mathbb{R}) \) on \( S_R \) by matrix multiplication on the left is called the geometric action.

Proposition 8.3. The set \( S_R \) is a principal homogeneous space for the geometric action.

Proof. Since \( N \in O_M(\mathbb{R}) \) preserves the form \( M \),

\[ (NW)^t G_M NW = W^t G_M W, \]

and so this action preserves \( S_R \). If \( R \) is invertible, then \( W \in S_R \) are invertible and so the element of \( O_M(\mathbb{R}) \) taking any \( W_1 \) to \( W_2 \), namely \( N := W_2W_1^{-1} \in O_M(\mathbb{R}) \), exists and is unique. That is, \( O_M(\mathbb{R}) \) is freely transitive on \( S_R \).

The reason for the name geometric is that, restricting to \( O_M^+(\mathbb{R}) \), the left action can be considered a Möbius action on circles via the exceptional isomorphism \( \rho \).

The special case of \( R = G_M \) gives \( S_R = O_M(\mathbb{R}) \).
Definition 8.4. Write $O_R(\mathbb{R})$ for the matrices preserving the quadratic form associated to Gram matrix $R$. The right action of $O_R(\mathbb{R})$ on $S_R$ by matrix multiplication on the right is called the algebraic action.

Proposition 8.5. The set $S_R$ is a principal homogeneous space for the algebraic action. Furthermore, $O_R(\mathbb{R}) \cong O_M(\mathbb{R})$.

Proof. Let $W_0 \in S_R$. Then $O_R(\mathbb{R}) = W_0^{-1} O_M(\mathbb{R}) W_0 \cong O_M(\mathbb{R})$ preserves the quadratic form given by Gram matrix $R$, which form is isomorphic to $M$ over $\mathbb{R}$. Therefore right multiplication by this group preserves $S_R$. The proof is as for the last proposition. □

The algebraic action is sometimes called the Apollonian action. It acts by linear combination on 4-tuples of vectors in $M$. It cannot be thought of as arising from an action on circles or vectors alone; it only acts on $S_R$.

The isomorphism between $O_R(\mathbb{R})$ and $O_M(\mathbb{R})$ given in Proposition 8.5 is called the algebraic-geometric correspondence depending on $W_0 \in S_R$, which we will denote

$$\sigma_{W_0} : O_M(\mathbb{R}) \to O_R(\mathbb{R}), \quad M \mapsto W_0^{-1} M W_0.$$ 

Both the geometric and algebraic actions have a manifestation in the action of Möbius. To discuss this, we have to interpret $S_R$ as clusters of circles.

Definition 8.6. Let $n \geq 4$, $n \in \mathbb{Z}$. A cluster type is a finite-to-one function

$$f : S_R \to \text{Circ}^n,$$

on a cluster space, with image lying in the collection of $n$-tuples of oriented circles, and having the form

$$f(W) = \left( \pi^{-1} \left( \sum_{i=1}^{4} a_{i,j} W_i \right) \right)^n_{j=1},$$

where $a_{i,j} \in \mathbb{Z}$ and $W_1, \ldots, W_4$ denote the columns of $W$. In other words, a cluster type determines a collection of circles by linear combination on the columns of $W$. A cluster type is given by the data of an invertible matrix $R$ and the $a_{i,j} \in \mathbb{Z}$, $1 \leq i \leq 4$, $1 \leq j \leq n$. A collection of circles in the image of a given cluster type is called an ordered cluster of the given type. An unordered cluster is any collection obtained from an ordered cluster by forgetting ordering.

The previous definition is motivated by the notion of a Descartes quadruple, which is the cluster type given by $n = 4$,

$$R = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix},$$

and $a_{i,j} = \delta_{i,j}$.

It is immediate to verify that under these definitions, the Möbius action corresponds to the geometric action, which justifies its name. The algebraic action can also be interpreted as an action of Möbius, via the algebraic-geometric correspondence. This is complicated by the need for a base cluster $W_0$ to define the algebraic-geometric correspondence, and by the need to consider ordered clusters as elements of Möbius. The precise statement is as follows, and is verified by direct computation.

Proposition 8.7. Fixing an ordered base cluster $B$ given by $W_B \in S_R$, the left- and right-multiplication action of Möbius on Möbius corresponds to the algebraic and geometric actions of $O_R(\mathbb{R})$ and $O_M(\mathbb{R})$, respectively, on $S_R$, in the sense that the following diagram commutes
The Apollonian structure of Bianchi groups

(where all arrows between sets are bijections, all arrows between groups are isomorphisms, and tightly curved arrows represent group actions):

In other words, an element of Möb can be interpreted as its image on the base cluster, and then the right multiplication action of Möb on Möb ‘is’ the algebraic action on clusters. When the right multiplication of Möb on Möb is interpreted in this way, we will simply refer to it as the algebraic action.

**Proposition 8.8.** The geometric and algebraic orbits of any cluster under any subgroup $G < \text{Möb}$ agree.

**Proof.** This follows from Proposition 8.7 by taking the given cluster as base cluster; then the orbits are both $G$ itself. □

9. APOLLONIAN GROUPS, WEAK AND STRONG, ALGEBRAIC AND GEOMETRIC

In this section we define the various flavours of Apollonian groups for a general imaginary quadratic field.

The limit set (or residual set) $\Lambda(G)$ of a subgroup $G \subset \text{Möb}$ is the accumulation set of the orbit of the origin.

**Definition 9.1.** A weak Apollonian group for the imaginary quadratic field $K \neq \mathbb{Q}(\sqrt{-3})$, or weak $K$-Apollonian group, is a finitely generated Kleinian group $A_K < \text{Möb}$ such that

1. Its limit set is the fundamental Apollonian packing for $K$, i.e. $\Lambda(A_K) = \mathcal{P}_K$.
2. $\mathcal{P}_K$ is a finite union of orbits of individual circles under $A_K$.

Recall that $\mathcal{P}_K$ denotes the fundamental packing (Definition 4.12). There is no reason to assume that there is a unique $K$-Apollonian group for a given field $K$. It is easy to give examples of weak $K$-Apollonian groups.

**Theorem 9.2.** Let $K$ be an imaginary quadratic field. Let $\mathcal{A}$ be the subgroup of Möb generated by $\text{PSL}_2(\mathbb{Z})$ and the matrix $V = \begin{pmatrix} 1 & \tau \\ 0 & -1 \end{pmatrix}$. Then $\mathcal{A}$ is a weak $K$-Apollonian group.

**Proof.** Since $\text{PSL}_2(\mathbb{Z})$ is finitely generated, so is $\mathcal{A}$. We use Proposition 4.6. Given a circle $M(\hat{\mathbb{R}})$, the immediately tangent circles are exactly those given by $M'(\hat{\mathbb{R}})$ where $M' \in M(\text{PSL}_2(\mathbb{Z}))V$. Therefore the orbit of $\hat{\mathbb{R}}$ includes all of the fundamental packing. Since we have $\text{PSL}_2(\mathbb{Z}) < \mathcal{A}$, all of $\hat{\mathbb{Q}}$ is in the orbit of $0$, so that $\hat{\mathbb{R}}$ is in the limit set $\Lambda(\mathcal{A})$, and so are all its images, i.e. all of $\mathcal{P}_K$. □

The proof illustrates that the fundamental packing is exactly the orbit of $\hat{\mathbb{R}}$ under $\mathcal{A}$. All other $K$-Apollonian packings are orbits of left cosets of $\mathcal{A}$.

**Theorem 9.3.** Any $K$-Apollonian packing is of Hausdorff dimension $\delta_K > 1$. 

whether the generators of $A_K$ of $\text{K}$

First, we show that $A_K$ is thin. For a finitely-generated Kleinian group, the limit set must be one of the following: totally disconnected, a circle, or of Hausdorff dimension $> 1$ (see, for example, [1, Corollary 1.8] and the citations therein). However, $P_K$, since it contains $\hat{\mathbb{R}}$ and other circles, is neither totally disconnected, nor a circle. \hfill \Box

Let $X$ be a subgroup of $H(\mathbb{Z})$, where $H$ is a semi-simple Lie group, and let $G = \text{Zcl}(X)$ be its Zariski closure in $H$. Then $X$ is called thin if it is of infinite index in $G(\mathbb{Z})$. We are interested in the case $H = O_M^+ \cong \text{Möb}$. We will consider a weak $K$-Apollonian group $\mathcal{A}$ to be a subgroup of $O_M^+$.

**Theorem 9.4.** Any weak Apollonian group $\mathcal{A}$ for $K$ is thin, and its Zariski closure is either $O_M$ or $SO_M$.

In any given situation, to determine which Zariski closure is obtained, it suffices to check whether the generators of $\mathcal{A}$ all lie in $SO_M$.

**Proof.** First, we show that $\mathcal{A}$ is of infinite index in $O_K(\mathbb{Z})$. For, its index is equal to the number of $K$-Apollonian packings in $S_K$. But there is a strip packing contained in every horizontal strip $k \leq 2m(z)/\sqrt{-\Delta} \leq k + 1$. Therefore there are infinitely many disjoint packings. On the other hand, $\mathcal{A}$ is infinite, since the infinite collection $\mathcal{P}_K$ of circles is a finite union of orbits of circles.

Let $\mathcal{G}$ be the Zariski closure of $\mathcal{A}$. It is necessarily an algebraic subgroup of $O_K$ defined over $\mathbb{Z}$. Therefore $\mathcal{G}(\mathbb{R})$ must be a Lie subgroup of $O_K(\mathbb{R})$. Our proof imitates that in [8, Lemma 1.6(ii)]. The classification of the a priori possibilities for $\mathcal{G}$ are:

1. A finite subgroup.
2. A torus or parabolic subgroup.
3. A subgroup fixing a form of signature $(1, 2)$.
4. $O_K$.
5. $SO_K$.

We eliminate the first three possibilities in turn. First, $\mathcal{G}(\mathbb{Z})$ is not finite as $\mathcal{A}$ is not finite. The second two possibilities are subgroups of dimension $\leq 2$. Any finitely generated Kleinian subgroup in those dimensions is geometrically finite. Therefore the limit set has Hausdorff dimension at most $1$. However, the residual set of $\mathcal{A}$ has Hausdorff dimension $> 1$, by Theorem 9.3. Therefore $\mathcal{G} = O_K$ or $SO_K$. In either case, as a subgroup of infinite index in $O_K(\mathbb{Z})$, $\mathcal{A}$ is of infinite index in $\mathcal{G}(\mathbb{Z})$. Therefore it is thin. \hfill \Box

We remark that the methods above also provide a proof that the subgroup $E_2(O_K)$ of $\text{PSL}_2(\mathbb{Z})$ generated by elementary matrices is thin whenever $O_K$ is non-Euclidean.

**Theorem 9.5.** When $O_K$ is non-Euclidean, the groups $E_2(O_K)$ are thin.

**Proof.** The group $E_2(O_K)$ is generated by $\text{PSL}_2(\mathbb{Z})$ and the matrix

$$W = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}.$$ 

First we show that $\Lambda(E_2(O_K))$ is the tangency-connected component of $\hat{\mathbb{R}}$ in $S_K$. Consider a circle given by $M(\hat{\mathbb{R}})$. Then the circles $MNW^*(\hat{\mathbb{R}})$, where $N \in \text{PSL}_2(\mathbb{Z})$ are exactly those tangent to $\hat{\mathbb{R}}$ in $S_K$ (see Proposition 3.7). Therefore the orbit of $\hat{\mathbb{R}}$ under $E_2(O_K)$ is the tangency-connected component of $S_K$ containing $\hat{\mathbb{R}}$. Since $\text{PSL}_2(\mathbb{Z}) < E_2(O_K)$, all of $\hat{\mathbb{R}}$ is in the limit set, so the entire tangency-connected component of $S_K$ is $\Lambda(E_2(O_K))$.

By [31, Theorem 7.1], there are infinitely many tangency-connected components in $S_K$. Hence $E_2(O_K)$ is of infinite index in $\text{PSL}_2(O_K)$. By the same method as Theorem 9.3, this
Theorem 9.8. The group \( \hat{A} \) has Hausdorff dimension exceeding 1. The rest of the proof of thinness is as in Theorem 9.4.

Next we define strong \( K \)-Apollonian groups, also called simply \( K \)-Apollonian groups. These are weak Apollonian groups with extra structure captured in their relationship to a certain cluster type. We are motivated by the traditional Apollonian group, which is described by its relationship to Descartes quadruples.

Definition 9.6. Let \( A \) be a Möbius and fix a cluster type. Suppose that

1. any two tangent circles in \( P_K \) are contained in infinitely many clusters inside \( P_K \),
2. via the Möbius action, the set of unordered clusters is a principal homogeneous space for \( A \).

Then we say that \( A \) is a strong \( K \)-Apollonian group with respect to the cluster type or simply a \( K \)-Apollonian group when no confusion will occur. If a base cluster is fixed, then \( A \) corresponds under the algebraic-geometric correspondence to a subgroup \( \hat{A} \) of \( O_K(\mathbb{Z}) \). In this case, \( \hat{A} \) is called an algebraic \( K \)-Apollonian group, and \( A \) is called a geometric \( K \)-Apollonian group when distinction is necessary.

The group \( \hat{A} \) acts on ordered clusters but not on unordered clusters or circles.

The terminology of strong and weak \( K \)-Apollonian groups requires justification.

Theorem 9.7. Let \( A \) be a Möbius, and fix a cluster type. Suppose \( A \) is a strong \( K \)-Apollonian group with respect to the cluster type. Then \( A \) is a weak \( K \)-Apollonian group.

Proof. Note that the second condition of Definition 9.6 guarantees that \( A \) takes circles in \( P_K \) to circles in \( P_K \).

First, we consider the limit set \( \Lambda(A) \). The limit set contains all images of 0 under \( A \). In particular, since \( P_K \) contains two circles tangent at 0, the assumptions of Definition 9.6 imply that it contains at least one tangency point in every \( K \)-cluster in \( P_K \).

Now, let \( z \) be a point of tangency within \( P_K \), between two circles \( C_1 \) and \( C_2 \). Then there are infinitely many \( K \)-clusters containing \( C_1 \) and \( C_2 \). In particular, since there are finitely many \( K \)-Bianchi circles of bounded curvature in any bounded region of \( \mathbb{C} \), there is a sequence of \( K \)-clusters containing \( C_1 \) and \( C_2 \) with curvatures approaching \( \infty \). Thus the other circles (besides \( C_1 \) and \( C_2 \)) of these clusters must be approaching \( z \) (since clusters are finite sets and tangency connected). Thus the collections of tangency points of the clusters are approaching \( z \), and so by the first observation, \( z \in \Lambda(A) \).

Since the tangency points on a circle of \( P_K \) are dense in the circle, every circle of \( P_K \) is contained in \( \Lambda(A) \), and so \( P_K \subset \Lambda(A) \).

For the converse, we use the fact that \( A \) takes circles of \( P_K \) to circles of \( P_K \), so that it must take \( \mathbb{R} \) only to circles in \( P_K \). Therefore, \( \Lambda(A) \subset P_K \).

For the second condition on an Apollonian group, it suffices to combine the three facts that

1. clusters are made up of exactly \( n \) circles,
2. every circle of \( P_K \) is contained in some cluster in \( P_K \), and
3. all clusters of \( P_K \) are in the orbit of any one cluster in \( P_K \).

\( \square \)

Theorem 9.8. The group \( A_{Q(i)} \) defined in Section 7 is a geometric \( Q(i) \)-Apollonian group, and \( \hat{A}_{Q(i)} \) is an algebraic \( Q(i) \)-Apollonian group.

Proof. It is known that there are infinitely many Descartes quadruples in an Apollonian circle packing containing two given circles. Theorem 7.2, via Proposition 8.8, tells us that the geometric action of \( A_{Q(i)} \) is transitive on the set of unordered Descartes quadruples. Furthermore, since the algebraic action is induced by the action of \( O_K(\mathbb{R}) \) on \( S_R \), we also know from Theorem
7.2 that there are no automorphisms of a cluster in $A_{Q(i)}$’s algebraic action, hence no elements fixing the base cluster in its geometric action. This proves that the set of unordered quadruples is a principle homogeneous space for $A_{Q(i)}$’s geometric action. □

A more general method of proving groups are Apollonian will be developed in Section 11.

10. Topographical Groups

For this section, we concern ourselves with certain special subgroups of $\text{PGL}_2(\mathbb{Z})$.

**Definition 10.1.** A superbasis is a triple $(a, b, c)$ of points of $\hat{\mathbb{Q}}$ which are pairwise distinct modulo all primes. A topographical group is a subgroup of $\text{PSL}_2(O_K)$ for which unordered superbases form a principal homogeneous space (under the usual Möbius action).

The use of the terminology superbasis is borrowed from Conway and Fung [4, The First Lecture]: for them, $a = [a_1, a_2], b = [b_1, b_2], c = [c_1, c_2] \in \mathbb{Z}^2$ form a superbasis if they are primitive vectors (i.e. $\gcd(a_1, a_2) = \gcd(b_1, b_2) = \gcd(c_1, c_2) = 1$), and each pair forms a basis for $\mathbb{Z}^2$. It is evident that the two definitions are naturally in bijection, where the vectors $a, b, c \in \mathbb{P}^1(\mathbb{Z})$ represent the elements $a = a_1/a_2, b = b_1/b_2, c = c_1/c_2 \in \hat{\mathbb{Q}}$.

Conway shows that the graph whose vertices are unordered superbases, where an edge indicates that two superbases share a basis, is a single tree of valence three, shown in Figure 12. Conway calls this graph the topograph.

A superbasis as above can be expressed as a $2 \times 2$ matrix with entries in $\mathbb{Z}$ given by

$$
\begin{pmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{pmatrix}
$$

where $c_1 = a_1 + b_1$ and $c_2 = a_2 + b_2$. The matrix expression has determinant $\pm 1$ and is unique up to multiplication of the matrix by $-1$ or reordering the superbasis. Therefore the set $\text{PGL}_2(\mathbb{Z})$ is in bijection with the set of ordered superbases. We write

$$
\phi : \text{PGL}_2(\mathbb{Z}) \rightarrow \{ \text{unordered superbases} \}
$$

equation for the map which forgets orientation.

Define

$$
S := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right\} < \text{PGL}_2(\mathbb{Z}).
$$

The group $S$ is isomorphic to $S_3$, and its elements, interpreted as ordered superbases, are all possible orderings of the superbasis $0, 1, \infty$. We have the following immediate characterisation.

**Proposition 10.2.** A subgroup $G < \text{PGL}_2(\mathbb{Z})$ is topographical if and only if $\text{PGL}_2(\mathbb{Z}) = G \times S$.

Furthermore, if $G$ is topographical, then $G$ acts on ordered superbases in exactly 6 orbits, each of which has exactly one ordering of each unordered superbasis.

Write $\Pi = \text{PGL}_2(\mathbb{Z})$ and $\Gamma = \text{PSL}_2(\mathbb{Z})$. Write $\Pi(N)$ and $\Gamma(N)$ for their congruence subgroups of level $N$, respectively.

**Theorem 10.3.** The only topographical groups are $G = \Gamma^3$ and $P = \Pi(2)$.

Note that these two groups are normal but not characteristic: the outer automorphism of $\text{PGL}_2(\mathbb{Z})$ maps one to the other (see [6, 33] for more on the outer automorphism). One is a congruence subgroup of $\text{PGL}_2(\mathbb{Z})$ and the other is not; this is a general phenomenon with regards to the outer automorphism [14].

**Proof.** From the classification of normal subgroups of small index in $\Gamma = \text{PSL}_2(\mathbb{Z})$ due to Newman [20], we immediately observe that if $G$ is topographical, then $G \cap \text{PSL}_2(\mathbb{Z})$ is one of $\Gamma(2), \Gamma', \Gamma^3$. 

where $\Gamma'$ represents the commutator subgroup, and $\Gamma(2)$ the congruence subgroup of level 2. The first two of these groups are of index 6 in $\text{PSL}_2(\mathbb{Z})$ and the latter is of index 3. All of these groups contain $\Gamma(12)$. But the only normal subgroups of $\text{PGL}_2(\mathbb{Z})$ of index 6 containing $\Gamma(12)$ are $G$ and $P$ (see [22]), whose intersections with $\Gamma$ are exactly $\Gamma^3$ and $\Gamma(2)$, respectively. □

**Theorem 10.4.** A topographical group is free on three generators of order 2 and with respect to these three generators, its Cayley graph under right multiplication is isomorphic to the topograph via $\phi$.

By the Cayley graph under right multiplication, it is meant that the edges $g$ and $gs$ (not $sg$) are joined for each generator $s$.

**Proof.** The relevant generators are

\[ G = \langle \gamma_1, \gamma_2, \gamma_3 \rangle, \]

where

\[ \gamma_1 = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]

and

\[ P = \langle \rho_1, \rho_2, \rho_3 \rangle, \]

where

\[ \rho_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}. \]

The generators $\gamma_i$ and $\rho_i$ are each of order 2. First we verify that they generate the groups given. If we set, as usual,

\[ S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \text{PGL}_2(\mathbb{R}), \]

then $\gamma_1 = TST^{-1}$, $\gamma_2 = T^{-1}ST$, and $\gamma_3 = S$. Since $\gamma_1 \gamma_2 \gamma_3 = T^3, \Gamma^3 < G$. But $\Gamma^3$ is normal, so it contains all conjugates of the element $S$, hence $G < \Gamma^3$. That the $\rho_i$ generate $\Pi(2)$ is immediate.

It is a computation to verify the isomorphism claimed. As a consequence, the groups are free products on their three generators. □

We will need the following lemma. Let

\[ B = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}, \]

which is the stabilizer of $\infty$.

**Lemma 10.5.**

\[ G \cap B = \langle \gamma_1 \gamma_2 \gamma_3 \rangle = \left\{ \begin{pmatrix} 1 & 3n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\} \cong \mathbb{Z}. \]

We will use the notation $G_B := G \cap B$.

**Proof.** We use the identification of the Cayley graph of $G$ under right multiplication with the topograph, as in the last theorem. One can draw the topograph in the plane so that the infinite ‘regions’ between branches are labelled with elements of $\hat{\mathbb{Q}}$, and the ‘shoreline’ consists of those superbases and bases containing that element, in the sense of Conway (see [4, The First Lecture] for details). The stabilizer of $\infty$ is exactly those elements of $G$ which map $\infty, 0, -1$ to a superbasis surrounding the region $\infty$, and keep $\infty$ in first position. Travelling along the ‘shore’ of the $\infty$ region by explicit computation, we see that this is exactly the group generated by $\gamma_1 \gamma_2 \gamma_3$. Finally,

\[ \gamma_1 \gamma_2 \gamma_3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}. \]
Figure 12. Conway’s topograph is a valence-three tree embedded in the plane such that it breaks up the plane into many infinite regions, each labelled with an element of $\mathbb{P}^1(\mathbb{Z})$. The vertices represent unordered superbases $a, b, c$ where $a, b, c$, are given by the three regions adjacent to the vertex. For example, the right-central vertex corresponds to the superbasis $0, 1, \infty$.

The structure of $\mathcal{A}_{\mathbb{Q}(i)}$ and the collection of quadruples has much in common with the structure of topographical groups and the collection of superbases. In particular, the collection of (unordered) Descartes quadruples can be considered the vertices of a graph, where edges indicate that two quadruples share a subset of three circles. This graph is a tree of valence four which is the Cayley graph of $\mathcal{A}_{\mathbb{Q}(i)}$ with respect to the four given generators. Each generator swaps out one circle of a quadruple, changing it into an adjacent quadruple. For more on this, see [30]. We will imitate this structure for other fields in later sections.

Finally, it is worth remarking that superbases can be interpreted as the tangency points of a triple of intervals covering $\hat{\mathbb{R}}$; such intervals can be put in bijection with elements of $\mathbb{R}^3$ lying on a hypersurface; this develops a story analogous to that of Section 6 with the exceptional isomorphism $O_{2,1}^+(\mathbb{R}) \cong \text{PGL}_2(\mathbb{R})$ in place of $O_{3,1}^+(\mathbb{R}) \cong \text{PGL}_2(\mathbb{C})$, and quadratic forms in place of Hermitian forms. This has pleasing connections to the rational parametrisation of Pythagorean triples.

11. SUFFICIENT CONDITIONS FOR APOLLONIAN GROUPS

In this section we provide a set of sufficient conditions guaranteeing a group is Apollonian. The main theorem, Theorem 11.4, will be used to verify all the example Apollonian groups in the remainder of the paper.

Let $\text{Möb}_{\hat{\mathbb{R}}}$ be the set of Möbius transformations fixing $\hat{\mathbb{R}}$. Define

$$\eta : \text{Möb}_{\hat{\mathbb{R}}} \to \text{PGL}_2(\mathbb{R})$$

to be the restriction of the action of such a Möbius transformation to $\hat{\mathbb{R}}$. This map is surjective.

We will need the following technical term for the theorem.
Definition 11.1. The base prong for $S_K$ is the collection of four $K$-Bianchi circles consisting of $\hat{\mathbb{R}}$ and three circles immediately tangent to $\hat{\mathbb{R}}$ at 0, 1 and $\infty$. A three-prong is a collection of four circles that has the same pairwise Pedoe products as the base prong. Equivalently, the three-prongs are the clusters in the cluster space containing the base prong. A three-prong is said to be centred on any of its circles which is tangent to the other three.

In the case of $\mathbb{Q}(i)$, three-prongs are exactly Descartes quadruples and a prong is centred on all of its circles. In other cases, the central circle is unique. Evidently, the base prong exists and is unique for a particular Schmidt arrangement $S_K$. The matrix $R$ for which the base prongs correspond to the cluster space $S_R$ is given explicitly in the next section, but is not needed at the moment.

Lemma 11.2. Three-prongs centred on $\hat{\mathbb{R}}$ are in bijection with superbases.

Proof. Any $K$-Bianchi circle tangent to $\hat{\mathbb{R}}$ at $\alpha/\beta$ (in lowest terms) is given by

$$\begin{pmatrix} \alpha & \gamma + \alpha k\tau \\ \beta & \delta + \beta k\tau \end{pmatrix}$$

where $\alpha\delta - \beta\gamma = 1$, $\alpha, \delta, \beta, \gamma \in \mathbb{Z}$, and where $k = 1$ if and only if the circle is immediately tangent (Proposition 3.7). Therefore its Pedoe embedding into $\mathbb{M}$ is given by

$$b = k(\tau - \tau)\beta^2i, \quad b' = k(\tau - \tau)\alpha^2i, \quad a = i(-(\alpha\delta - \beta\gamma) + k(\tau - \tau)\alpha\beta) = i(k(\tau - \tau) - 1).$$

Then the Pedoe product of two $K$-Bianchi circles tangent to $\hat{\mathbb{R}}$ at $\alpha_1/\beta_1$ and $\alpha_2/\beta_2$ (in lowest terms), is

$$1 + \frac{1}{2}k_1k_2(\tau - \tau)^2(\alpha_1\beta_2 - \alpha_2\beta_1)^2$$

where $k_1, k_2 \in \mathbb{Z}$. Therefore, we obtain

$$1 + \frac{1}{2}(\tau - \tau)^2$$

if and only if $\alpha/\beta$ and $\gamma/\delta$ satisfy $\alpha\delta - \beta\gamma = \pm 1$ and the circles are immediately tangent. The result follows. \[\square\]

We will also need a notion of automorphisms of a cluster.

Definition 11.3. The automorphisms of a cluster are the M"{o}bius transformations taking the cluster back to itself. The images of one circle under these automorphisms are its automorphs.

Automorphisms preserve Pedoe products, as do all M"{o}bius transformations. Therefore automorphisms take three-prongs within a cluster to three-prongs within the cluster.

Theorem 11.4. Let $\mathcal{A} < \text{M"{o}b}$. Suppose that

(I) $\mathcal{A}$ takes circles in $\mathcal{P}_K$ to circles in $\mathcal{P}_K$, and
(II) $\eta(\mathcal{A})$ contains a topographical group.

Suppose there is a cluster type, and a cluster (call it the base cluster), for which the following conditions hold:

(i) The base cluster is contained in $\mathcal{P}_K$.
(ii) The base cluster is the unique cluster containing the base prong,
(iii) The automorphisms of the base cluster permute the three circles tangent to $\hat{\mathbb{R}}$ at 0, 1 and $\infty$ transitively.
(iv) There is an element of $\mathcal{A}$ taking $\hat{\mathbb{R}}$, or one of its automorphs within the base cluster, to the circle immediately tangent to $\hat{\mathbb{R}}$ at 0.
(v) No automorphism of the base cluster is contained in $\mathcal{A}$.

Then, $\mathcal{A}$ is an Apollonian group for $\mathbb{K}$. 

Proof. We will verify each of the conditions of Definition 9.6.

First, we will show that all clusters are in the orbit of the base cluster. Since \( \mathcal{A} \) is a group, the orbit of the base cluster under the geometric (left multiplication) or algebraic (right multiplication) actions is the same (see Proposition 8.7). We will focus on the algebraic action.

To begin, we will show that any cluster in \( \mathcal{P}_K \) which contains a three-prong centred on \( \hat{R} \) is in the orbit of the base cluster. It suffices to observe that

1. the three-prongs centred on \( \hat{R} \) are in bijection with superbases (Lemma 11.2),
2. the three-prongs centred on \( \hat{R} \) are also in bijection with clusters in \( \mathcal{P}_K \) containing such three-prongs (this is by assumptions (i) and (ii), and the Möbius action), and
3. the action of \( \mathcal{A} \) on \( \hat{R} \) is transitive on superbases, by assumption (II).

Let \( C \) be the circle immediately tangent to \( \hat{R} \) at 0. By assumption (iv) on the base cluster, and the last observation, there is, in the orbit of the base cluster, a cluster containing a three-prong centred on \( C \). But then by the last observation, and assumption (iii) on the base cluster, we see that in the orbit of the base cluster there is a cluster containing a three-prong centred on any circle immediately tangent to \( \hat{R} \).

By this method we see that in fact for any circle \( C \) in \( \mathcal{P}_K \), since it is obtained from \( \hat{R} \) by a finite chain of immediate tangencies, there is in the orbit of the base cluster, a cluster containing a three-prong centred on \( C \). Since every cluster contains a three-prong centred on some circle (to see this, note that the base cluster does by assumption, and that all clusters have the same pairwise Pedoe products), every cluster is in the orbit of the base cluster. Hence all clusters are in the orbit of any cluster, by the algebraic action.

The second condition of Definition 9.6 now follows from assumption (v).

Now we verify the first condition of Definition 9.6, that is, that any two tangent circles in \( \mathcal{P}_K \) are contained in infinitely many clusters. Of course, it suffices, by the Möbius action, to prove this for \( \hat{R} \) and \( C \), tangent at 0. It suffices to observe that there are infinitely many superbases containing 0. \( \square \)

In the following sections we define example Apollonian groups for each imaginary quadratic field. These definitions are not unique, and the author has endeavoured to find pleasing choices.

12. \( K \)-clusters for imaginary quadratic fields

In this section, we develop a general theory that yields an Apollonian group for any imaginary quadratic \( K \) (save \( \mathbb{Q}(\sqrt{-3}) \), as usual). In the Euclidean cases, it is possible to replace this group with one which is freely generated by elements of order two, and this is the purpose of later sections. We begin, however, taking \( K \) generally.

**Definition 12.1.** Given a set, \( D \), of four oriented circles corresponding to vectors \( v_1, \ldots, v_4 \) in \( M \), define the matrix \( W_D \) formed of the columns

\[
\begin{bmatrix}
v_1, v_2, v_3, v_4
\end{bmatrix}
\]

in the case that \( \Delta \equiv 0 \pmod{4} \) and

\[
\begin{bmatrix}
v_1, \frac{1}{2}(-v_2 + v_3 + v_4), \frac{1}{2}(v_2 - v_3 + v_4), \frac{1}{2}(v_2 + v_3 - v_4)
\end{bmatrix}
\]

in the case that \( \Delta \equiv 1 \pmod{4} \).
We say that $C_1, \ldots, C_4$ form a $K$-cluster, or $K$-quadruple, if $W_D$ satisfies the relationship $W_D^2 G_M W_D = R$, where the matrix $R$ is defined\(^5\) for $\Delta \equiv 0 \pmod{4}$ to be

$$R_0 := \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 + \frac{1}{2}\Delta & 1 + \frac{1}{2}\Delta \\ -1 & 1 + \frac{1}{2}\Delta & 1 & 1 + \frac{1}{2}\Delta \\ -1 & 1 + \frac{1}{2}\Delta & 1 + \frac{1}{2}\Delta & 1 \end{pmatrix},$$

and for $\Delta \equiv 1 \pmod{4}$ to be

$$R_1 := \begin{pmatrix} 1 & -1/2 & -1/2 & -1/2 \\ -1/2 & 1 & 1 + \frac{1}{2}\Delta & 1 + \frac{1}{2}\Delta \\ -1/2 & 1 + \frac{1}{2}\Delta & 1 & 1 + \frac{1}{2}\Delta \\ -1/2 & 1 + \frac{1}{2}\Delta & 1 + \frac{1}{2}\Delta & 1 \end{pmatrix}.$$

These two cases are united in the sense that in the case $\Delta \equiv 1 \pmod{4}$, we have simply made the convenient change of variables indicated by (9) and (10). In other words, $R_0 = S^t R_1 S$ where $S$ is the change of variables

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

This is done to simplify the generators in the next definition.

In particular, a $K$-cluster always consists of a three-prong as defined in the last section, and $S_R$ is the cluster space of three-prongs. In the case $\Delta \neq -4$, these three circles are disjoint from

\(^5\)The inverses of $R_0$ and $R_1$ are, respectively, $\frac{1}{\Delta} \begin{pmatrix} \Delta + 3 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$ and $\frac{1}{\Delta} \begin{pmatrix} \Delta + 3 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{pmatrix}$. In particular, that tells us the “Descartes equation” for $K$-clusters: the curvatures $a, b, c, d$ of four circles in a $K$-cluster satisfy $2(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2 - (\Delta + 4)a^2 = 0$. 

Figure 13. The base $K$-cluster, here shown for $\mathcal{S}_{\mathbb{Q}(\sqrt{-15})}$. 

The Apollonian structure of Bianchi groups
one another, as in Figure 13. The definition above reduces to the usual Descartes quadruple (where they are all mutually tangent) when $\Delta = -4$.

**Definition 12.2.** We define the matrix group $\overline{A}_K \subset O_R(\mathbb{R})$ to be the group generated by the following generators. For the case $\Delta \equiv 0 \pmod{4}$:

$$
\begin{pmatrix}
1 & 2 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 2 & 0 & 1 \\
0 & 2 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 2 & 0 \\
0 & 0 & 2 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 2 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 2 \\
0 & 0 & 1 & 2 \\
0 & 1 & 0 & 2 \\
0 & 0 & 0 & -1
\end{pmatrix},
\begin{pmatrix}
0 & 1 + \Delta/4 & 1 & 1 + \Delta/4 \\
0 & 1 & 0 & 0 \\
1 & -1 - \Delta/4 & 0 & -1 - \Delta/4 \\
0 & 0 & 0 & 1
\end{pmatrix},
$$

(11)

while for the case $\Delta \equiv 1 \pmod{4}$:

$$
\begin{pmatrix}
1 & -1 & 1 & 1 \\
0 & -1 & 2 & 2 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 1 & -1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 2 & -1 & 2 \\
0 & 1 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 1 & 1 & -1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 2 & 2 & -1
\end{pmatrix},
\begin{pmatrix}
0 & 1 & \Delta + 3 & 0 \\
1 & 1 & -\frac{\Delta + 1}{4} & -1 \\
0 & 0 & 1 & 0 \\
1 & 0 & -\frac{\Delta + 3}{4} & 0
\end{pmatrix}.
$$

(12)

We wish to associate to this a group of Möbius transformations via the algebraic-geometric correspondence. Define the base cluster $D$ to be given by the images of $\mathbb{R}$ under

$$
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
-1 & \tau \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & -1 \\
-1 & \tau - 1
\end{pmatrix},
\begin{pmatrix}
1 & 1 - \tau \\
1 & -\tau
\end{pmatrix}.
$$

The associated matrix $W_D$ to this $K$-cluster is, for $\Delta \equiv 0 \pmod{4}$,

$$
W_D = W^0_D := \begin{pmatrix}
0 & 0 & \eta & \eta \\
0 & \eta & 0 & \eta \\
0 & 0 & 0 & \eta \\
-1 & 1 & 1 & 1
\end{pmatrix},
$$

and for $\Delta \equiv 1 \pmod{4}$,

$$
W_D = W^1_D := \begin{pmatrix}
0 & \eta & 0 & 0 \\
0 & 0 & \eta & 0 \\
0 & \eta/2 & \eta/2 & -\eta/2 \\
-1 & 1/2 & 1/2 & 1/2
\end{pmatrix},
$$

where $\eta = i(\tau - \tau) = \sqrt{\Delta}$. Note that $W^0_D = W^1_D S$.

The four generators of $\overline{A}_K$, under the algebraic-geometric correspondence via the base quadruple, become

$$
\begin{pmatrix}
-1 & 2 \\
-1 & 1
\end{pmatrix},
\begin{pmatrix}
1 & -1 \\
2 & -1
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 1 - \tau \\
0 & -1
\end{pmatrix}.
$$

(13)

The reader will recognise the first three matrices as $\gamma_1, \gamma_2, \gamma_3$. The last matrix takes $\mathbb{R}$ to the circle tangent to $\mathbb{R}$ at $\infty$. The group generated by these Möbius transformations will be called $A^0_K$. Note that $A^0_{Q(i)} \neq A^0_{Q(i)}$.

**Theorem 12.3.** Suppose $\Delta \leq -15$. Then $A^0_K$ (or $\overline{A}_K$) has the presentation

$$(s_1, s_2, s_3, r : s_1^2 = s_2^2 = s_3^2 = r^2 = 1, rs_1s_2s_3 = s_3s_2s_1r).$$

In fact, this presentation may be realised by taking $s_1, s_2, s_3, r$ to be the matrices (11) or (12), depending on whether $\Delta \equiv 0 \pmod{4}$.

To prove this, we need a result from the Bass-Serre theory of groups acting on trees (in our case, the tree will be a tangency-connected component of the graph of immediate tangencies of $\mathcal{P}$).
Theorem 12.4 (Serre [29, Section 1.4]). Suppose that a group $G$ acts on a tree $X$ with inversion (i.e. there exists $g \in G$ and $e$ an edge of $X$ such that $g \cdot e$ is again $e$, but with orientation reversed). Suppose the action is transitive on vertices and transitive on edges. Let $v$ be a vertex of $X$ and $e$ be an adjacent edge. Let $G_v$ be the stabilizer of $v$, $G_e$ be the stabilizer of $e$, and let $G'$ be the stabilizer of $v$ and $e$ (hence preserving the direction of $e$). Then $G'$ is of index two in $G_e$ and

$$G \cong G_v *_{G_e} G_e,$$

where $*$ denotes the free product with amalgamation.

Proof of Theorem 12.3. The group $A_K$ is isomorphic to a subgroup of Möbius transformations given by generators (13), which we will call $\Omega$. As Möbius transformations act on circles, preserving tangencies, and $\Omega$ takes a $K$-Apollonian packing $P$ back to itself, it acts on the tangency tree of $P$. The fourth generator of (13), call it $r$, reverses the edge between circle $\mathbb{R}$ and $\mathbb{R} + \tau$. Note that the topographical group $G$ is a subgroup of $\Omega$.

The orbit of $rG$ acting on $\mathbb{R}$ is the collection of $K$-Bianchi circles immediately tangent to $\mathbb{R}$. Thus $\Omega$ maps $\mathbb{R}$ to all immediately tangent circles. Therefore the action is transitive on vertices. The orbit of $\infty$ under $G$ is $\hat{Q}$. This implies that the action is transitive on edges (which correspond to tangency points).

The stabilizer of $\infty$ is $(G_B, r)$ (see Lemma 10.5 for the definition of $G_B$). Since conjugation by $r$ acts as the non-trivial automorphism of $G_B \cong \mathbb{Z}$ (a simple direct computation), we have that this stabilizer is isomorphic to the non-trivial semi-direct product $\mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z})$. The stabilizer of $\hat{R}$ is $G$. The stabilizer of the directed edge $\hat{R} \rightarrow_{\infty} (\hat{R} + \tau)$ is $G_B$.

With these data, Theorem 12.4 describes the structure of $A_K$ as $G *_{G_B} \langle G_B, r \rangle$, and the presentation follows from Lemma 10.5. \qed

Theorem 12.5. Suppose $\Delta < -15$. Then the group $A_K^0$ is a geometric Apollonian group for $K$, and $A_K$ is an algebraic Apollonian group for $K$.

Proof. This proof uses Theorem 11.4, so that it suffices to verify a few details about the base cluster and $A_K^0$. In particular, it was noted above that $G < A_K^0$. All the needed verifications save (v) are immediate. For (v), we must appeal to the tangency tree. In particular, the automorphisms of the base cluster must stabilize $\hat{R}$; this stabilizer is exactly $G$. But $G$ contains no automorphisms of superbases, so there are no automorphisms of the base cluster in $A_K^0$. \qed

13. Cubes and cubicles in $K = \mathbb{Q}(\sqrt{-2})$

In the case of $K = \mathbb{Q}(\sqrt{-2})$, there is a very pretty Apollonian group, and we will elaborate on it somewhat here.

In the graph of tangencies, we find cubes, as Figure 14. A cube is defined to be eight circles in the following arrangement specified by the Pedoe products. Here $W_C$ is a $4 \times 8$ matrix which has as columns the eight $\pi(C_i)$:

$$W_C^t G_M W_C = \begin{pmatrix}
1 & -1 & -3 & -1 & -5 & -3 & -1 & -3 \\
-1 & 1 & -1 & -3 & -3 & -5 & -3 & -1 \\
-3 & -1 & 1 & -1 & -3 & -5 & -3 & -1 \\
-1 & -3 & -1 & 1 & -3 & -1 & -3 & -5 \\
-5 & -3 & -1 & 3 & 1 & -1 & -3 & -1 \\
-3 & -5 & -3 & -1 & 1 & -1 & -3 & -1 \\
-1 & -3 & -5 & -3 & -3 & -1 & 1 & -1 \\
-3 & -1 & -3 & -5 & -1 & -3 & -1 & 1
\end{pmatrix}$$

(14)
This matrix represents the following arrangement of tangencies:

\[
\begin{array}{cccc}
\v_1 & \v_2 \\
\v_4 & \v_3 \\
\v_7 & \v_8 \\
\v_6 & \v_5 \\
\end{array}
\]

It is evident that a cube contains eight \(\mathbb{Q}(\sqrt{-2})\)-clusters as defined in the previous section, corresponding to the eight vertices of the cube (considered up to reordering the three circles tangent to a central one). Any such \(\mathbb{Q}(\sqrt{-2})\)-cluster defines a unique cube (this follows from verifying the fact for the base cluster).

We define a cubicle to be a subset of four circles of a given cube so that no two are tangent. There are two cubicles in a cube. A cubicle satisfies this arrangement:

\[
\begin{bmatrix}
1 & -3 & -3 & -3 \\
-3 & 1 & -3 & -3 \\
-3 & -3 & 1 & -3 \\
-3 & -3 & -3 & 1 \\
\end{bmatrix}
\]

(15)

A cubicle is contained in a unique cube. By considering a single cubicle in \(\mathbb{Q}(\sqrt{-2})\) (we use Figure 14), one can obtain equations that determine the cube from the cubicle:

\[
\begin{align*}
2v_2 &= v_1 + v_3 - v_6 + v_8, \\
2v_4 &= v_1 + v_3 + v_6 - v_8, \\
2v_5 &= -v_1 + v_3 + v_6 + v_8, \\
2v_7 &= v_1 - v_3 + v_6 + v_8.
\end{align*}
\]

Any individual face of a cube consists of four \(\mathbb{Q}(\sqrt{-2})\)-Bianchi circles satisfying the following relations (in particular, they are tangent in a loop):

\[
\begin{bmatrix}
1 & -1 & -3 & -1 \\
-1 & 1 & -1 & -3 \\
-3 & -1 & 1 & -1 \\
-1 & -3 & -1 & 1 \\
\end{bmatrix}
\]

(16)

Then there are exactly two ways to complete a face to a cube of \(\mathbb{Q}(\sqrt{-2})\)-Bianchi circles (again, this need only be verified on the base cube). This implies that given a cube, there are six swaps one can perform which fix one side and swap out the remaining four circles. Geometrically, this is accomplished by reflecting in the circle \(C\) orthogonal to all circles in the fixed side, as shown in Figure 15.

These six swaps can be described in terms of the cube as follows. Fixing \(v_1, \ldots, v_4\), the two ways to complete to a cube containing this side are to add \(v_5, \ldots, v_8\) or \(v_5', \ldots, v_8'\) where

\[
\begin{align*}
v_5 + v_5' &= -2v_1 + 4v_2 + 4v_4 \\
v_6 + v_6' &= 4v_1 - 2v_2 + 4v_3 \\
v_7 + v_7' &= 4v_2 - 2v_3 + 4v_4 \\
v_8 + v_8' &= 4v_1 + 4v_3 - 2v_4
\end{align*}
\]

Incidentally, this gives a relation on the curvatures of a cubicle: \(8(a^2 + b^2 + c^2 + d^2) = 3(a + b + c + d)^2\) which we do not need here.

It is also nice to observe that all body diagonal sums agree: \(v_1 + v_5 = v_2 + v_6 = v_3 + v_7 = v_4 + v_8\), and that on each individual face, diagonal sums again agree: \(v_1 + v_3 = v_2 + v_4\) etc.
Figure 14. Base tent for $\mathcal{S}_{\mathbb{Q}(\sqrt{-2})}$. The coordinates of the eight circles in $M$ are given, in the labelled order, by the columns of the following matrix:

$$
\begin{pmatrix}
0 & 2\sqrt{2} & 2\sqrt{2} & 4\sqrt{2} & 4\sqrt{2} & 2\sqrt{2} & 2\sqrt{2} \\
0 & 4\sqrt{2} & 4\sqrt{2} & 4\sqrt{2} & 2\sqrt{2} & 2\sqrt{2} & 2\sqrt{2} \\
-1 & 3 & 3 & 1 & 5 & 3 & 1 & 3
\end{pmatrix}.
$$

The two cubicles forming the cube are shown in blue and black.

Figure 15. The blue and black circles (including two straight lines) form the base cube of the $\mathbb{Q}(\sqrt{-2})$-Apollonian packing containing $\hat{\mathbb{R}}$. If swapping through the side represented by black circles, the blue circles are replaced with green ones, by inversion in the red circle. The new cube is formed of the black and green circles.
If we express a cube as a real $4 \times 4$ matrix whose columns are the cubicle $v_1, v_3, v_6, v_8$, then these swaps correspond to right multiplication by

$$
\begin{pmatrix}
1 & 0 & 3 & 3 \\
0 & 1 & 3 & 3 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
\end{pmatrix},
\begin{pmatrix}
1 & 3 & 0 & 3 \\
0 & 0 & 0 & -1 \\
0 & 3 & 1 & 3 \\
0 & -1 & 0 & 0 \\
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 3 \\
3 & 0 & 1 & 3 \\
-1 & 0 & 0 & 0 \\
\end{pmatrix},
\begin{pmatrix}
1 & 3 & 3 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 3 & 3 & 1 \\
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
3 & 3 & 0 & 1 \\
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
3 & 1 & 3 & 0 \\
3 & 3 & 1 & 0 \\
\end{pmatrix}.
$$

(17)

Definition 13.1. Call the group generated by these matrices $\hat{A}_{Q(\sqrt{-2})}$.

The Möbius transformations which realise the six swaps on the base cube of Figure 14 are given by (in the same order as the generators (17)),

$$
z \mapsto \frac{\bar{\tau}}{2\pi - 1},
\begin{aligned}
z & \mapsto -\bar{\tau} + 2, \\
z & \mapsto \frac{(3 + 2\sqrt{-2})\bar{\tau} - 4}{4\pi - 3 + 2\sqrt{-2}},
\end{aligned}
\begin{aligned}
z & \mapsto \frac{(1 + 2\sqrt{-2})\bar{\tau} - 2}{4\pi - 1 + 2\sqrt{-2}}, \\
z & \mapsto \frac{(1 + 2\sqrt{-2})\bar{\tau} - 4}{2\pi - 1 + 2\sqrt{-2}}.
\end{aligned}
$$

(18)

Definition 13.2. Call the group generated by these transformations $A_{Q(\sqrt{-2})}$.

Definition 13.3. The cube graph is the graph whose vertices are all cubes (considered without regard to the ordering of the constituent circles); and whose edges indicate cubes sharing a side.

Theorem 13.4. The cube graph is a forest of trees of valence 6. Consequently, $A_{Q(\sqrt{-2})}$ is freely generated by the generators (17). In particular, it is a free product of six copies of $\mathbb{Z}/2\mathbb{Z}$.

Proof. Let $H$ be the cube graph. Any one component of the graph of ordered cubes under the six available swaps is the Cayley graph of $A_{Q(\sqrt{-2})}$ under right multiplication. We wish to demonstrate that any component of $H$ is a tree. If so, then the Cayley graph cannot have a loop, as it would reduce to a loop in the cube graph under forgetting orientation. The theorem follows.
Let $G$ be the tangency graph of a single packing, i.e. the graph whose vertices are circles, and whose edges represent tangencies. This graph can be embedded on the sphere $\hat{C}$ by placing each vertex at the center of the corresponding circle, so that edges pass through tangency points (the exception being the bounding circle, where we can take its center to be $\infty$). Therefore it is planar.

Suppose that $H$ has a loop. This loop is a finite loop of $n \geq 2$ cubes lying in one packing, so its circles create an induced subgraph $L$ of $G$ (for that packing). We will demonstrate that $L$ cannot exist inside the planar graph $G$.

To do so, we will consider building the graph $L$ cube-by-cube. Begin with one adjacent pair of cubes in $L$, sharing four vertices in a cycle $C$. The cycle breaks the plane up into two regions. The two cubes lie in the two different regions (except they both include the boundary). (By the Möbius action, verification on the base cube suffices.) Continuing to add cubes to form $L$, at each stage we are adding edges and vertices of $G$ inside an existing face of the construction. In particular, we are either adding a cube inside the cycle $C$ or outside it. Finally, to complete the loop $L$, we must join one cube inside $C$ to one outside $C$, using another cycle $C' \neq C$. This is impossible inside the planar graph $G$. $\square$

An alternate method of proof is to show that $L$ contains $4n$ vertices and $8n$ edges, but no cycles of length 3, which is impossible for a planar graph. Interestingly, a similar count works for $Q(i)$ and $Q(\sqrt{-7})$, but fails for $Q(\sqrt{-11})$. The proof above generalises to all cases with little modification.

**Theorem 13.5.** The group $A_{Q(\sqrt{-7})}$ is a geometric Apollonian group for $Q(\sqrt{-2})$ and $\hat{A}_{Q(\sqrt{-2})}$ is an algebraic Apollonian group for $Q(\sqrt{-2})$.

**Proof.** Our cluster type is that given above (cubes), and the base cube is as in Figure 14. We will verify the hypotheses of Theorem 11.4. Each of these verifications is a straightforward computation or has been verified in the foregoing material in this section. In particular, the topographical group contained in $A_{Q(\sqrt{-7})}$ is $P$; it is given by the first, second and fourth generators of (18). Furthermore, the automorphs of any vertex of the cube include all other vertices. Finally, that no automorphism of the base cluster is contained in $A_{Q(\sqrt{-7})}$ is a consequence of the proof of Theorem 13.4: such an automorphism would indicate a nontrivial path in the Cayley graph whose endpoints were two different orderings of the same cube. Forgetting orientation, this would give a loop in the cube graph. $\square$

14. Tents and belts in $K = Q(\sqrt{-7})$

In this case, and for $Q(\sqrt{-11})$, we will be somewhat more brief in our description. We define a $Q(\sqrt{-7})$-Descartes configuration, called a tent, to be five circles in the arrangement specified by the relation

$$W_D G_M W_D = \begin{pmatrix}
1 & -1 & -5/2 & -1 & -5/2 \\
-1 & 1 & -1 & -5/2 & -1 \\
-5/2 & -1 & 1 & -1 & -5/2 \\
-1 & -5/2 & -1 & 1 & -1 \\
-5/2 & -1 & -5/2 & -1 & 1
\end{pmatrix},$$

on the matrix $W_D$ whose columns are the five circles. The graph of tangencies looks like this:
Figure 17. Base tent for $S_8(\sqrt{7})$. The coordinates of the five circles in $M$ are given, in the labelled order, by the columns of the following matrix:

\[
\begin{pmatrix}
0 & 0 & \sqrt{7} & \sqrt{7} & \sqrt{7} \\
\sqrt{7} & 0 & \sqrt{7} & \sqrt{7} & 0 \\
0 & 0 & \sqrt{7} & \sqrt{7}/2 & 0 \\
1 & -1 & 1 & \sqrt{7}/2 & 1 \\
\end{pmatrix}
\]

Circles tangent to three others in the tent are shown in black, while those tangent to two others are shown in blue.

There is one relation among these circles:

$$v_1 + v_3 + v_5 = 2(v_2 + v_4).$$

Three of the five circles ($v_1, v_3, v_5$) are distinguished as being tangent to fewer other circles in the tent (2 instead of 3). Call such a circle a peak. Removing a peak leaves four circles in a cycle, called a belt. Given a belt, there are exactly two tents containing it. Given a belt $v_1, v_2, v_3, v_4$, the two peaks $v_5$ and $v'_5$ that complete it to a tent satisfy

$$v_5 + v'_5 = v_1 + v_2 + v_3 + v_4.$$

The other peak $v'_5$ is tangent to $v_1$ and $v_3$, thus:

\[
\begin{array}{c}
v_1 \\ v_2 \\ v'_5 \\ v_3 \\ v_4
\end{array}
\]

Therefore $v_2$ and $v_4$ become new peaks. It is appropriate to renumber the resulting tent so we have

$$v'_1 = v_2, \quad v'_2 = v_1, \quad v'_4 = v_4, \quad v'_3 = v_3, \quad v'_5 = v_1 + v_2 + v_3 + v_4 - v_5.$$

An example is shown in Figure 18.
Figure 18. The green and black circles (including two straight lines) form the base tent of the \( \mathbb{Q}(\sqrt{-7}) \)-Apollonian packing containing \( \hat{R} \). If swapping out the peak given by the green circle, the green circle is replaced with the blue one. The new tent is formed of the black and blue circles. Note that the black circles are not individually fixed; they are permuted.

\[
\frac{1 + \sqrt{7}i}{2}
\]

Figure 19. The \( \mathbb{Q}(\sqrt{-7}) \)-Apollonian packing containing \( \hat{R} \).

We will write a tent as a 4 \times 4 matrix whose columns are \( v_1, v_2, v_3, v_4 \); call this a tentbase. Four circles form a tentbase if and only if the matrix of their columns, \( W_D \), satisfies

\[
W_D^T G_M W_D = \begin{pmatrix}
1 & -1 & -5/2 & -1 \\
-1 & 1 & -1 & -5/2 \\
-5/2 & -1 & 1 & -1 \\
-1 & -5/2 & -1 & 1
\end{pmatrix} =: R.
\]

There are three moves that will replace a tent with another that shares a belt: each of the three peaks can be swapped out. These correspond to multiplying the tentbase on the right by these
three matrices of order two:

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix},
\begin{pmatrix}
-2 & 0 & 0 & -1 \\
3 & 0 & 1 & 2 \\
0 & 1 & 0 & -1 \\
0 & -1 & -2 & 0 \\
\end{pmatrix},
\begin{pmatrix}
0 & -1 & 0 & 1 \\
0 & 2 & 3 & 0 \\
3 & 0 & 0 & 2 \\
1 & 2 & 3 & 0 \\
\end{pmatrix}.
\]

(19)

Definition 14.1. Denote the subgroup of \(O_R(\mathbb{R})\) generated by the matrices (19) by \(\hat{A}_{Q(\sqrt{-7})}\).

The three swaps (19) are not realised as inversions in circles as in the case of \(Q(i)\) and \(Q(\sqrt{-2})\). On the base tent of Figure 17, they correspond, via the algebraic-geometric correspondence, to the following M"obius transformations, respectively:

\[
z \mapsto \frac{1 + \sqrt{-7}}{2} z + \frac{1 - \sqrt{-7}}{2}, \quad z \mapsto -z - \frac{3 - \sqrt{-7}}{2}, \quad z \mapsto \frac{z}{1 - \frac{1}{\sqrt{-7}} z - 1}.
\]

(20)

Definition 14.2. The 7-tent graph is the graph whose vertices are all tents (considered without regard to ordering); and whose edges indicate tents sharing a belt.

Theorem 14.3. The 7-tent graph is a forest of trees of valence 3. Consequently, \(A_{Q(\sqrt{-7})}\) is freely generated by the generators (19). In particular, it is a free product of three copies of \(Z/2Z\).

Proof. The proof is exactly as in Theorem 13.4: any two tents lie in the two regions created by their shared belt.

\[\square\]

Theorem 14.4. The group \(A_{Q(\sqrt{-7})}\) is a geometric Apollonian group for \(Q(\sqrt{-7})\) and the group \(\hat{A}_{Q(\sqrt{-7})}\) is an algebraic Apollonian group for \(Q(\sqrt{-7})\).

Proof. One can verify the hypotheses of Theorem 11.4 directly, exactly as in the proof of Theorem 13.5, with the exception of (II), which does not hold. Therefore, we wish to use a slightly modified version of the proof of Theorem 11.4. The part of the proof that relies on (II) is the statement that any cluster in \(P_K\) containing a three-prong centred on \(\hat{R}\) is in the orbit of the base cluster. It suffices to show that each superbasis adjacent to \((0,1,\infty)\) in the topograph is achieved. Composing each pair of generators (20) in each possible order, we obtain 6 transformations. It is a computation to verify that, applying these 6 transformations to the base cluster, we obtain among the results clusters containing the three-prongs on \((0,1/2,1)\), \((1,2,\infty)\) and \((-1,0,\infty)\), namely the three superbases adjacent to \((0,1,\infty)\). In this way we modify the proof of Theorem 11.4 to reach the needed conclusion.

\[\square\]

15. TENTS AND BELTS IN \(K = Q(\sqrt{-11})\)

This case is similar to the case \(Q(\sqrt{-7})\). A tent \(D\) consists of 10 circles, and contains four belts (loops of tangent circles) of 6 circles each, in the following arrangement:
Let $W_D$ denote the matrix whose columns are the vectors in $M$ corresponding to these circles, say $v_i$, $i = 1, \ldots, 10$ of $M$. We have


The ten circles of a tent span a vector space of dimension 4. A presentation of the relations is:

$$\begin{align*}
5v_2 &= -4v_1 - 4v_3 + v_5 + v_7 \\
5v_4 &= v_1 - 4v_3 - 4v_5 + v_7 \\
5v_6 &= -4v_1 + v_3 - 4v_5 + v_7 \\
5v_8 &= v_1 - 4v_3 + v_5 - 4v_7 \\
5v_9 &= v_1 + v_3 - 4v_5 - 4v_7 \\
5v_{10} &= -4v_1 + v_3 + v_5 - 4v_7
\end{align*}$$

A tent contains four belts. Within a belt, the sum of opposite circles is invariant, e.g.

$$v_1 + v_4 = v_2 + v_5 = v_3 + v_6.$$ 

Since there are four belts, one obtains four vectors; these are independent, and we therefore represent a tent as a matrix with these four columns, say

$$v_1 + v_4, v_1 + v_8, v_1 + v_9, v_3 + v_9.$$ 

There are exactly two tents containing a single belt. Therefore there are four swaps one can perform. Each swap preserves one belt and changes three belts. An example is shown in Figure 21. The resulting matrices, in terms of the representation above, are

$$\begin{pmatrix} 1 & 3 & 3 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 3 & 1 & 3 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 3 & 3 & 1 & 3 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 3 & 3 & 3 & 1 \end{pmatrix}.$$ 

Note that the circles of the belt are permuted; for example, the first of these has the effect

$$v_1' = v_4, v_2' = v_5, v_3' = v_6, v_4' = v_1, v_5' = v_2, v_6' = v_3.$$ 

**Definition 15.1.** Denote the subgroup of $O_R(\mathbb{R})$ generated by these four matrices by $A_{\mathbb{Q}(\sqrt{-1})}$.

The four Möbius maps performing these swaps on the base tent of Figure 20 (in the same order as (21)) are:

$$z \mapsto \frac{3 + \sqrt{-11}z - 2}{3z + \sqrt{-11}}, \quad z \mapsto \frac{1 + \sqrt{-11}z - 2}{2z + \sqrt{-11}}, \quad z \mapsto \frac{3 + \sqrt{-11}z - 3}{2z + \sqrt{-11}}, \quad z \mapsto \frac{2 + \sqrt{-11}z - 4}{4z - 2 + \sqrt{-11}}.$$ 

**Definition 15.2.** Denote the subgroup of Möb generated by these four matrices by $A_{\mathbb{Q}(\sqrt{-11})}$.

**Definition 15.3.** The 11-tent graph is the graph whose vertices are all tents (considered without regard to ordering); and whose edges indicate tents sharing a belt.

**Theorem 15.4.** The 11-tent graph is a forest of trees of valence 4. Consequently, $A_{\mathbb{Q}(\sqrt{-11})}$ is freely generated by the generators (21). In particular, it is a free product of four copies of $\mathbb{Z}/2\mathbb{Z}$. 


Figure 20. Base tent for $S_{\mathbb{Q}(\sqrt{-11})}$. The coordinates of the ten circles in $\mathcal{M}$ are given, in the labelled order, by the columns of the following matrix:

$$
\begin{pmatrix}
0 & 0 & \sqrt{11} & 2\sqrt{11} & 2\sqrt{11} & \sqrt{11} & 2\sqrt{11} & 2\sqrt{11} & 3\sqrt{11} & \sqrt{11} \\
0 & \sqrt{11} & 2\sqrt{11} & \sqrt{11} & 2\sqrt{11} & \sqrt{11} & 2\sqrt{11} & 2\sqrt{11} & 0 \\
0 & 0 & \sqrt{11}/2 & 3\sqrt{11}/2 & 3\sqrt{11}/2 & \sqrt{11}/2 & \sqrt{11}/2 & \sqrt{11}/2 & 3\sqrt{11}/2 & 0 \\
-1 & 1 & 9/2 & 13/2 & 9/2 & 1 & 9/2 & 13/2 & 13/2 & 1
\end{pmatrix}
$$

Circles tangent to three others in the tent are shown in black, while those tangent to two others are shown in blue.

Figure 21. The green and black circles (including two straight lines) form the base tent of the $\mathbb{Q}(\sqrt{-11})$-Apollonian packing containing $\hat{R}$. If swapping out the peak given by the green circles, the green circles are replaced with the blue ones. The new tent is formed of the black and blue circles. Note that the black circles are not individually fixed; they are cyclically permuted.
The Apollonian structure of Bianchi groups

Figure 22. The $\mathbb{Q}(\sqrt{-11})$-Apollonian packing containing $\hat{\mathcal{R}}$.

Table 1. For each discriminant $\Delta$, the table shows the smallest power of 2, $M$, that explains the obstructions at the prime 2, and the observed residue sets $S_M$ modulo $M$.

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>$M$</th>
<th>$S_M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \equiv 0, 4, 16 \pmod{32}$</td>
<td>2</td>
<td>$S_2 = {0, 1}, {1}$</td>
</tr>
<tr>
<td>$\Delta \equiv 8, 24 \pmod{32}$</td>
<td>4</td>
<td>$S_4 = {0, 2, 3}, {0, 1, 2}$</td>
</tr>
<tr>
<td>$\Delta \equiv 12 \pmod{32}$</td>
<td>4</td>
<td>$S_4 = {0, 1}, {1, 2}, {2, 3}, {0, 3}$</td>
</tr>
<tr>
<td>$\Delta \equiv 20 \pmod{32}$</td>
<td>4</td>
<td>$S_4 = {1}, {3}, {0, 1, 2, 3}$</td>
</tr>
<tr>
<td>$\Delta \equiv 28 \pmod{32}$</td>
<td>8</td>
<td>$S_8 = {0, 1, 4}, {2, 3, 6, 7}, {0, 4, 5}$</td>
</tr>
<tr>
<td>otherwise</td>
<td>none</td>
<td>none</td>
</tr>
</tbody>
</table>

Table 2. For each discriminant $\Delta$, the obstruction at the prime 3 is explained by the first power, 3. The table shows the observed residue sets $S_3$ modulo 3.

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>$S_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \equiv 5, 8 \pmod{12}$</td>
<td>$S_3 = {0, 1}, {0, 2}$</td>
</tr>
<tr>
<td>otherwise</td>
<td>none</td>
</tr>
</tbody>
</table>

Proof. The proof is exactly as in Theorem 13.4 and Theorem 14.3. □

Theorem 15.5. The group $A_{\mathbb{Q}(\sqrt{-11})}$ is a geometric Apollonian group for $\mathbb{Q}(\sqrt{-11})$, and the group $\hat{A}_{\mathbb{Q}(\sqrt{-11})}$ is an algebraic Apollonian group for $\mathbb{Q}(\sqrt{-11})$.

Proof. The proof is very similar to Theorem 14.4. □

16. Curvatures in $K$-Apollonian packings

This section is devoted to some computational data supporting Conjecture 1.4. We first record a basic result on curvatures.

Theorem 16.1. Let $\mathcal{P}$ be a $K$-Apollonian circle packing. Then the reduced curvatures of $\mathcal{P}$ have no common factor.
Proof. \( \mathcal{P} \) is generated by a circle \( C \) of curvature \( b \in \mathbb{Z} \) under immediate tangency. By [31, Theorem 4.7], the curvatures of the circles immediately tangent to \( C \) (these circles are in \( \mathcal{P} \)) are

\[
N(x) - b
\]

for \( x \) in a rank two \( \mathbb{Z} \)-lattice \( \beta \mathbb{Z} + \delta \mathbb{Z} \subset \mathcal{O}_K \). But \( \beta \) and \( \delta \) are coprime, as they form the lower row of a matrix of unit determinant. Therefore the norms of \( \beta, \delta \) and \( \beta + \delta \) cannot share a common factor (any rational prime \( p \) splits into at most two distinct primes in \( \mathcal{O}_K \), and if \( p \mid N(x) \) then \( x \in \mathfrak{p} \) for some \( p \mid \mathfrak{p} \)). Therefore the reduced curvatures in \( \mathcal{P} \) cannot all share a common factor. \( \square \)

Using Sage Mathematics Software [32], some computer experiments were performed. The author computed the complete set of reduced curvatures in various \( K \)-Apollonian packings modulo various moduli \( n \). The results suggest Conjecture 1.4. In particular, each Apollonian packing was observed to omit certain modular equivalence classes. The behaviour of individual primes is independent, so that we can discuss the obstruction at a prime \( p \) as all equivalence classes modulo powers of \( p \) that cannot occur. The obstruction is explained by \( p^k \) if \( p^k \) is the largest power of \( p \) needed to describe the obstruction. In experiments, the only obstructions that occurred were at 2 (always explained by 2, 4 or 8) and at 3 (always explained by 3). This explains the number 24 in Conjecture 1.4. Tables 1 and 2 give the observed allowable sets of residues. It is conjectured that these tables are complete.

References


