General Zelevinsky Algebras

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Abstract

Here we show that a fairly general notion of 'Hopf algebra with positivity and self-adjointness' admits a decomposition into a tensor product of 'atomic' such objects. Then, for an infinite family of base rings, we classify the atomic objects. This generalizes both a theorem of Zelevinsky, which he applied to linear representations of various families of groups, and an analogous theorem of Bean and Hoffman, which had applications to projective representations of symmetric and alternating groups (and more generally to some covering groups of monomial groups). The sequence of examples for which the classification is complete starts with these two earlier results. The remaining terms of the sequence will apply to representations of covers of certain wreath product groups.

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If $R(S_n)$ denotes the free abelian group with basis the isomorphism classes of irreducible complex linear representations of the symmetric group (where a positive linear combination is thought of as the representation which is the corresponding direct sum of irreducibles), then the direct sum of these groups has quite a bit of extra structure. For example, it is a *graded ring*, where multiplying a pair of representations (of S_n and S_m) is done by first forming a tensor product (a representation of $S_n \times S_m$), followed by inducing to S_{n+m}. Reversing this, using restriction instead of induction, gives a *comultiplication.* The third piece of extra structure is simply the specifying of a particular basis over $\mathbf{Z}$, in this case, the *irreducibles.* Zelevinsky **[Z]** abstracted this, and applied it to the representation theory of several other sequences of groups. Bean and Hoffman**[B-H]** introduced a variation in which the ground ring was changed to something a bit bigger than $\mathbf{Z}$ (see Section 5 below). They proved theorems analogous to Zelevinsky's, which both showed how to decompose such an object as a tensor product, and which classified the indecomposables. See **[H-H]** for applications to representations of several families of groups as well as to the projective representations of the symmetric and alternating groups.

In Section 4 below, we do the decomposition when the ground ring is very general, delineated by a few axioms. A family of new examples of this, given in Section 5, is then classified completely by determining the indecomposables in Section 6.

For the proof of decomposition in Section 4, we follow well known arguments in Hopf algebra theory and the methods initiated by Zelevinsky. While reading the rather lengthy definitions immediately below, the reader may find it helpful to refer to the examples in Section 5.

1. Definitions.

Let G be an abelian group with operation denoted $+$.

The Base Ring L.

Let $\{ L_g \ : \ g \in G \}$ be a collection of abelian groups indexed by G. Either this collection, or $\bigoplus_G L_g$, would be referred to as a *graded abelian group* (graded over G). We'll have little or no need to refer to non-homogeneous elements of such a direct sum. Let

$$L_g \otimes_{\mathbf{Z}} L_h \ \longrightarrow \ L_{g+h}$$

be a collection of group morphisms which, when collected together, would make $\bigoplus_G L_g$ into a commutative ring (with $1 \in L_0$). So we now have a graded commutative ring L.

Let $\{ \mathcal{B}_g \subset L_g \ : \ g \in G \}$ be $\mathbf{Z}$-bases for the groups L_g, which are therefore free abelian groups. By a *positive element* of L, we shall mean any linear combination of $\mathcal{B} := \bigcup_G \mathcal{B}_g$ in which the coefficients are non-negative integers. Assume that products of elements from $\mathcal{B}$ are strictly (i.e. non-zero) positive.

Finally, suppose given a map $\rho : G \to \mathcal{B}_0$, which maps onto a subgroup of invertibles in L, and, as such, is actually a morphism of groups. In particular, we are assuming that $1 \in \mathcal{B}_0$.

It follows that,
(i) a sum of positive elements is positive, and is non-zero as long as at least one of the summands is non-zero;
(ii) a product of positive elements is positive, and is non-zero as long as all of them are non-zero (although L is not an integral domain in many applicable cases);
(iii) if $\sum_\alpha \ell_\alpha^2 = 1$ with ℓ_α being positive elements, then all but one of the ℓ_α are zero.

The Tensor Factor Switching "Sign".

Let $\mathbf{N}$ denote the additive semi-group of non-negative integers. Suppose given a map

$$\epsilon \ : \ (G \times \mathbf{N}) \ \times \ (G \times \mathbf{N}) \ \longrightarrow \ G$$

which is

1) *bi-additive*—i.e. a morphism of additive semi-groups with respect to each factor $(G \times \mathbf{N})$ when the element in the other factor is fixed; and

2) *both antisymmetric and symmetric*—i.e. $\epsilon(y,\ x) \ = \ -\epsilon(x,\ y) = \ \epsilon(x,\ y)$ for x and y in $G \times \mathbf{N}$. (So ϵ takes values in the 2-torsion subgroup of G.)

Whenever we have two $(G \times \mathbf{N})$-graded L-modules C and B, the switch map, τ, will be the morphism of modules determined by specifying it on 'homogeneous pure tensors' as follows:

$$\tau \ : \ B \otimes_L C \ \rightarrow \ C \otimes_L B$$

$$b \otimes c \ \mapsto \quad \nu(b,c)\, c \otimes b \ \text{ where } \nu(b,c) = \rho\epsilon[(g_1,k_1),(g_2,k_2)] \ \text{ for } b \in B_{g_1,k_1} \text{ and } c \in C_{g_2,k_2} \, .$$

L-PSH algebras.

These are the main objects of study. To begin we are given an indexed collection of abelian groups $\{\ A_{g,k}\ :\ g \in G,\ k \in \mathbf{N}\ \}$, plus group morphisms

$$L_h \otimes_{\mathbf{Z}} A_{g,k} \ \longrightarrow \ A_{g+h,k}$$

which, for each $k \in \mathbf{N}$, makes

$$A_k := \bigoplus_{g \in G} A_{g,k}$$

into a G-graded module over L. The next piece of extra structure is a collection of module morphisms

$$\mu = \{\mu_{k,\ell} \ : \ A_k \otimes_L A_\ell \ \rightarrow \ A_{k+\ell}\} \quad \text{and} \quad \eta \ : \ L \rightarrow A$$

which, collected together, would make $\oplus_k A_k$ into an L-algebra, with η an isomorphism onto A_0, and the image of 1_L under η acting as an identity element. It doesn't matter whether we assume associativity of this multiplication, since that will follow from the other structure and axioms (see **2.2**).

Next assume given module morphisms in the opposite directions to the above,

$$\Delta = \{\Delta_{k,\ell} \ : \ A_{k+\ell} \ \rightarrow \ A_k \otimes_L A_\ell\} \quad \text{and} \quad \beta \ : \ A \rightarrow L \ ,$$

with the properties as with multiplication (expressed via arrows) but with all arrows reversed. Assume that $\beta|_{A_k}$ is zero for positive k, and is the inverse of η (with reduced codomain) for $k = 0$. Again, coassociativity can be deduced later.

Finally, assume that the following diagram commutes; it says that Δ is a morphism of algebras (since the definition given in **3** below of the multiplication on a tensor product uses the twisted switch map). It also says that μ is a morphism of coalgebras.

$$\begin{array}{ccccc}
A \otimes A & & \xrightarrow{\quad\quad\quad \mu \quad\quad\quad} & & A \\
\Delta \otimes \Delta \downarrow & & & & \downarrow \Delta \\
A \otimes A \otimes A \otimes A & \xrightarrow[1 \otimes \tau \otimes 1]{} & A \otimes A \otimes A \otimes A & \xrightarrow[\mu \otimes \mu]{} & A \otimes A
\end{array}$$

Ignoring the fact that τ is not necessarily the usual flip map (but nearly is), these properties are those which express the fact that we now have a connected graded *bi-algebra* A. It is a *Hopf algebra* because, again from the extra structure, we'll sketch below in **2.4** the construction of a 'canonical' conjugation (whose existence in the ungraded theory must be postulated to distinguish Hopfs among the bi-'s).

Next comes the positivity structure. Assume also given a collection

$$\{ \ I_{g,k} \subset A_{g,k} \ : \ g \in G, \ k \in \mathbf{N} \ \}$$

of sets such that $\rho(G)I_{g,k} \subset I_{g,k}$. (The phrase 'set of special irreducibles' motivates using I, at the risk of confusion with 'ideal', which it is not.) This gives an action of G on $I_{g,k}$, a *cross-section* of which means a choice of one element in each orbit of the action. Assume also that some (and therefore any) cross-section $X_{g,k}$ has the property that $X_k := \cup_g X_{g,k}$ is an L-basis for the L-module $A_k := \bigoplus_g A_{g,k}$. So A is free as an L-module. It follows that A is a free abelian group with basis $\{ \ bx \ : \ b \in \mathcal{B}, \ x \in X \ \}$. We shall also assume that $X_{0,0} = \{1\}$ whenever a cross-section is chosen.

By a *positive element* of A we mean any linear combination from $X := \bigcup_k X_k$ in which the coefficients are positive elements of L. We'll have no need to consider inhomogeneous such elements. A positive element of $A \otimes_L A$ will mean the same thing, this time using the basis $\{ \ y \otimes z \ : \ y, \ z \in X \ \}$. These definitions are independent of which cross-section X is chosen in $I := \bigcup_{g,k} I_{g,k}$.

The two main axioms connecting these structures are the following.

P. *The maps μ, Δ, η and β all map positive elements to positive elements.*

S. *There is a choice of cross-section X, which determines L-valued inner products as below, with respect to which the pairs $(\mu, \ \Delta)$ and $(\eta, \ \beta)$ are both adjoint pairs (so A is a self-dual Hopf algebra) :*

On A, use the inner product $< \, , >$ for which the basis X is orthonormal. On $A \otimes_L A$, do the same thing, but with the basis $\{ \ y \otimes z \ : \ y, \ z \in X \ \}$. These inner products are not independent of the choice of X, and different choices can affect whether the last phrase in axiom S holds. But if the non-identity elements in $\rho(G)$ are all of order 2 (as happens with all the examples in **5** below), then the inner products are independent of choice of X. In general, the two inner products are symmetric, and are related by the identity

$$< a \otimes b, c \otimes d > = < a, c >< b, d > \quad .$$

Note that there is no 'sign' here associated with moving b past c. The inner product in L is clear: $< 1, 1 > := 1$.

In A, a sum of positive elements is positive, and is non-zero as long as at least one of the summands is non-zero. This is easy to see and will be used quite often below. Furthermore, given a positive $a \in A$ such that $< a, a >= 1$, it follows that there is an invertible $\ell \in L$ such that $\ell a \in I$. Prove this using (iii) in the definition of positivity in L. (In fact, one gets that $\ell a \in X$ where $\ell^2 = 1$.)

Note that there is no question of the *consistency* of the axioms for an L-PSH-algebra, for any fixed L. We can take A_0 to be L itself, and $A_k = 0$ for $k > 0$, with the unique structure as a Hopf algebra over itself, and with $I = \rho(G)$. In the main result below, **4.2**, decomposing A as a tensor product of 'atoms', this is the degenerate instance where there are no atoms at all.

It won't be necessary to define 'map of L-PSH-algebras' in general. An *isomorphism* is the obvious thing : an isomorphism of Hopf algebras which preserves gradings and preserves the sets I. (This makes sense with respect to the application to representations— i.e. I is part of the structure, but X is not, in any natural way.) A *sub-L-PSH-algebra* B of A is an L-PSH-algebra which is a graded sub-Hopf algebra of A, and is such that $I_B = B \cap I_A$.

2. Basic results.

Here we show how associativity and a form of commutativity (and their duals), as well as the existence of the Hopf algebra conjugation, can be deduced from the other assumptions.

Let A be any L-PSH-algebra. Borrowing from old fashioned algebraic topology, denote $\bigoplus_{k>0} A_k$ as $\tilde{A}$. Let

$$\tilde{A}^2 \ := \ \mathrm{Span}_L\{ \ bc \ : \ b, c \in \tilde{A} \ \} \ .$$

A *primitive* element c in a coalgebra is one for which $\Delta(c) \ = \ 1 \otimes c \ + \ c \otimes 1$. Let P be the set of all primitive elements.

Lemma 2.1. $\tilde{A}^2 \cap P \ = \ \{0\}$.

Proof. If $p \in P$ and $b, c \in \tilde{A}$, then, by adjointness,

$$< p \ , \ bc \ > = < \ \Delta p \ , \ b{\otimes}c \ > = < \ p{\otimes}1{+}1{\otimes}p \ , \ b{\otimes}c \ > = < \ p \ , \ b \ > 0 + 0 < \ p \ , \ c \ > = \ 0 \, .$$

But $< x, x >= 0$ implies that $x = 0$.

Theorem 2.2. *The algebra A is necessarily associative—and so, by self-duality, it is co-associative as a coalgebra.*

Proof. Since $A_0 \cong L$, we need consider only homogeneous elements in $\tilde{A}$. Let

$$\zeta(x,y,z) \;:=\; (xy)z - x(yz) \;.$$

This is clearly in $\tilde{A}^2$ for x, y and z in $\tilde{A}$. To show that it is zero, we apply **2.1**, after showing by induction on the sum of the three $\mathbf{N}$-degrees that it is in P. Writing, in the manner of Hopf algebraists, $\Delta(x) \;=\; \sum x' \otimes x''$, and similarly for y and z, and using the trilinearity of ζ, and since Δ commutes with multiplication,

$$\Delta\zeta(x,y,z) \;=\; \zeta(\Delta x, \Delta y, \Delta z) \;=\; \sum \zeta(x' \otimes x'',\; y' \otimes y'',\; z' \otimes z''\,) \;.$$

Now, recalling $\nu(c,d) := \rho\epsilon[\text{grading}(c),\ \text{grading}(d)]$, the term inside the last summation is

$$\nu(x'',y')\;[(x'y')\otimes(x''y'')](z'\otimes z'') \;-\; \nu(y'',z')\;(x'\otimes x'')[(y'z')\otimes(y''z'')]$$

$$=\; \nu(x'',y')\nu(y'',z')\nu(x'',z')\;\{[(x'y')z']\otimes[(x''y'')z''] \;-\; [x'(y'z')]\otimes[x''(y''z'')]\}\;.$$

When $\zeta(x',y',z') = 0$, this last { difference } becomes $(x'y'z')\otimes\zeta(x'',y'',z'')$; and similarly with primes and double primes interchanged. But, by the inductive hypothesis, one of these holds for all terms, and both hold for all but two terms (showing them to be zero). Those two terms then give

$$(1\cdot 1\cdot 1)\otimes\zeta(x,y,z) \;+\; \zeta(x,y,z)\otimes(1\cdot 1\cdot 1)\;,$$

showing that $\zeta(x,y,z)$ is primitive, as required.

With respect to the maps ρ and ϵ of the previous section, we define an algebra B graded over $(G \times \mathbf{N})$ to be *$\rho\epsilon$-commutative* if $\mu \circ \tau = \mu$, that is, if

$$bc \;=\; \nu(b,c)cb \qquad\qquad \text{for all homogeneous } b \text{ and } c \text{ in } B\;.$$

The notion of *$\rho\epsilon$-cocommutativity* of a coalgebra C is defined by requiring that $\tau \circ \Delta = \Delta$. These two definitions can be obtained from each other by reversing the arrows in the diagrams which express them arrow-theoretically.

Theorem 2.3. *The algebra A is $\rho\epsilon$-commutative—and so, by self-duality, it is $\rho\epsilon$-cocommutative as a coalgebra.*

Proof. Here one can simply mimic the proof of **2.2**, but changing ζ to the two-variable function

$$\zeta(x,y) \;:=\; xy \;-\; \nu(x,y)yx\;.$$

The only difference is the calculation checking that

$$\zeta(x',y')=0 \quad\Longrightarrow\quad \zeta(x'\otimes y'\ ,\ x''\otimes y'') \;=\; (x'y')\otimes\zeta(x'',y'')$$

and

$$\zeta(x'',y'')=0 \quad\Longrightarrow\quad \zeta(x'\otimes y'\ ,\ x''\otimes y'') \;=\; \zeta(x',y')\otimes(x''y'')\ .$$

For example, to do the first of these, one needs the identity

$$\nu(x',y'')^{-1}\,\nu(x,y) \;=\; \nu(x',y')\,\nu(x'',y')\,\nu(x'',y'')\ .$$

All the elements here are, of course, homogeneous.

Theorem 2.4. *The algebra A admits a unique map $\gamma : A \to A$ which is an endomorphism of a $G\times\mathbf{N}$-graded module such that the following diagram is commutative:*

$$\begin{array}{ccccc} A\otimes A & & \xrightarrow{\quad 1\otimes\gamma \quad} & & A\otimes A \\ \Delta\uparrow & & & & \downarrow\mu \\ A & \xrightarrow[\beta]{} & L & \xrightarrow[\eta]{} & A \end{array}$$

Furthermore,
(1) the same diagram, except replacing $1\otimes\gamma$ by $\gamma\otimes 1$, also commutes; and
(2) the map γ is an automorphism of algebras; that is,

$$\gamma(uv) \;=\; \gamma(u)\gamma(v)\ .$$

Proof. The set of graded module morphisms, from an $\mathbf{N}$-graded coalgebra, C, to an $\mathbf{N}$-graded algebra, B, forms a group under the operation $*$, where $\delta*\lambda := \mu_B\circ(\delta\otimes\lambda)\circ\Delta_C$. The usual proof works here, where we have the extra grading over G. The group identity element is $\eta\circ\beta$. (The main work is defining, by induction on the $\mathbf{N}$-grading of the element, the value on an element of the group inverse of a morphism.) This proves the first part, by taking B=A=C, and observing that we're simply asking γ to be the right (group) inverse of the identity map of A.

Then (i) holds since γ will also be a left inverse. [The above group is in general commutative when both B and C are, by a straightforward argument. In our case, the same argument works even with the twist in the definition of τ, which produces $\rho\epsilon$-commutativity in A, but strict commutativity in the group. That's a second proof of (i).]

One next checks that $\gamma_{A\otimes B} \;=\; \gamma_A\otimes\gamma_B$ by a straightforward diagram chase. For (ii), we want $\mu\circ(\gamma\otimes\gamma) \;=\; \gamma\circ\mu$. This follows by diagram chases which show that both of these are the group inverse for μ (taking C to be $A\otimes A$, and B to be A, in the definition of the group above). For $\gamma\circ\mu$, this is straightforward. For $\mu\circ(\gamma\otimes\gamma)$, it is not entirely straightforward, partly because it depends on $\gamma_{A\otimes B} \;=\; \gamma_A\otimes\gamma_B$ with $B=A$, but also because it depends on showing that $\mu\circ(\mu\otimes\mu)\circ(1\otimes\tau\otimes 1) \;=\; \mu\circ(\mu\otimes\mu)$. It suffices to check this when applied to a pure tensor $u\otimes v\otimes w\otimes x$. The right-hand side gives $uvwx$, and the left-hand side gives $\nu(v,w)uwvx$, so this follows from $\rho\epsilon$-commutativity.

3. Tensor Products of Algebras.

Here we recall what is meant by an 'infinite' tensor product, since the main theorem on decomposition, **4.2**, doesn't require a finiteness hypothesis. If $\{A_\alpha\}$ is an indexed collection of algebras with identity element, then there is a direct system of algebras $\bigotimes_{\alpha\in F} A_\alpha$ (with injective algebra maps) indexed by finite subsets F of the index set, directed by set inclusion, as follows: whenever $F_1 \subset F_2$, map $\bigotimes_{\alpha\in F_1} A_\alpha$ to $\bigotimes_{\alpha\in F_2} A_\alpha$ by sending the 'pure tensor' $\otimes_{\alpha\in F_1} a_\alpha$ to the same pure tensor except that 1_{A_α} is also entered in slots for which $\alpha \in F_2 \setminus F_1$. Then $\bigotimes_\alpha A_\alpha$ is defined to be the direct limit of this system. To be more concrete and to check the existence of the limit, we can write any element of the limit (non-uniquely) as a finite L-linear combination of the finite pure tensors as above, where the usual multilinear non-uniqueness is supplemented by the ability to enter or erase tensor factors equal to identity elements in any pure tensor. This infinite tensor product has the usual universal mapping property with respect to 'infi-linear' maps from a suitably restricted Cartesian product of the A_α.

Theorem 3.1. *Any tensor product of L-PSH-algebras is again an L-PSH-algebra, where the structure for a tensor product is specified as follows.*

For two L-PSH-algebras A and B, the L-module $A \otimes_L B$ is the usual, but the multiplication and comultiplication are defined using the '$\rho\epsilon$-twist' τ :

$$\mu_{A\otimes B} \; : \; A\otimes B\otimes A\otimes B \xrightarrow{1\otimes\tau\otimes 1} A\otimes A\otimes B\otimes B \xrightarrow{\mu_A\otimes\mu_B} A\otimes B$$

$$\Delta_{A\otimes B} \; : \; A\otimes B \xrightarrow{\Delta_A\otimes\Delta_B} A\otimes A\otimes B\otimes B \xrightarrow{1\otimes\tau\otimes 1} A\otimes B\otimes A\otimes B$$

Define the unit map as the composite

$$\eta_{A\otimes B} \; : \; L \xrightarrow{\cong} L\otimes_L L \xrightarrow{\eta_A\otimes\eta_B} A\otimes B \; .$$

The counit comes by reversing the arrows.
The positivity is defined by $I_{A\otimes B} := \{ \, x\otimes y \; : \; x\in I_A, \; y\in I_B \, \}$. (Here we can have $x\otimes y = x'\otimes y'$ when $x \neq x'$ and $y \neq y'$.)

For finitely many factors, just iterate this, using the natural associativity of $\otimes$.

For infinitely many factors, any finite set of elements can be thought of as lying in a finite tensor product, where the previous tells us what to do.

Proof. The proof for more than two factors is only more complicated in notation, so we'll give it for two. Let A and B be two L-PSH-algebras. That the structure defined above gives a graded Hopf algebra is proved in the usual straightforward way, the twisted shift map here not causing any complications. If X and Y are cross-sections for A and B, then the set $\{ \, x\otimes y \; : \; x\in X, \; y\in Y \, \}$ is easily checked to be one for the tensor product, and that's certainly an L-basis. Checking axiom P is simple calculation. To prove axiom S, check the adjointness conditions on pure tensors involving homogeneous elements, and appeal to linearity. The longer of the two calculations is for adjointness of

$(\mu_{A\otimes B}\ ,\ \Delta_{A\otimes B})$, as follows. Since six different inner products are involved, we have put subscripts on the notations for them.

$$
\begin{aligned}
<\ a\otimes b\ ,\ (a'\otimes b')(a''\otimes b'')\ >_{A\otimes B} &= <\ a\otimes b\ ,\ \nu(b',a'')\ (a'a'')\otimes(b'b'')\ >_{A\otimes B} \\
&= \nu(a'',b') < a\ ,\ a'a'' >_A < b\ ,\ b'b'' >_B \\
&= \nu(a'',b') <\ \Delta_A(a)\ ,\ a'\otimes a''\ >_{A\otimes A} <\ \Delta_B(b)\ ,\ b'\otimes b''\ >_{B\otimes B}\ .
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&<\ \Delta_{A\otimes B}(a\otimes b)\ ,\ a'\otimes b'\otimes a''\otimes b''\ >_{A\otimes B\otimes A\otimes B} \\
&\qquad = <\ (1\otimes\tau\otimes 1)[\Delta_A(a)\otimes\Delta_B(b)]\ ,\ a'\otimes b'\otimes a''\otimes b''\ >_{A\otimes B\otimes A\otimes B} \\
&\qquad \overset{(*)}{=}\ \nu(b',a'') <\ \Delta_A(a)\ ,\ a'\otimes a''\ >_{A\otimes A} <\ \Delta_B(b)\ ,\ b'\otimes b''\ >_{B\otimes B}\ ,
\end{aligned}
$$

as required. Equality $(*)$ follows from multilinearity by checking on pure tensors $x = a_1\otimes a_2$ and $y = b_1\otimes b_2$, and letting $x = \Delta_A(a)$ and $y = \Delta_B(b)$:

$$
\begin{aligned}
&<\ (1\otimes\tau\otimes 1)(x\otimes y)\ ,\ a'\otimes b'\otimes a''\otimes b''\ >_{A\otimes B\otimes A\otimes B} \\
&\qquad = \nu(a_2,b_1) <\ a_1\otimes b_1\otimes a_2\otimes b_2\ ,\ a'\otimes b'\otimes a''\otimes b''\ >_{A\otimes B\otimes A\otimes B} \\
&\qquad = \nu(a'',b') <\ a_1\ ,\ a'\ >_A <\ b_1\ ,\ b'\ >_B <\ a_2\ ,\ a''\ >_A <\ b_2\ ,\ b''\ >_B \\
&\qquad = \nu(a'',b') < a_1\otimes a_2\ ,\ a'\otimes a'' >_{A\otimes A} < b_1\otimes b_2\ ,\ b'\otimes b'' >_{B\otimes B} \\
&\qquad = \nu(a'',b') < x\ ,\ a'\otimes a'' >_{A\otimes A} < y\ ,\ b'\otimes b'' >_{B\otimes B}\ .
\end{aligned}
$$

4. Decomposition Theorem.

Recall that a *primitive* element c in a coalgebra is one for which $\Delta(c) = 1\otimes c + c\otimes 1$. Let X be any cross-section in an L-PSH-algebra A, and let P be its set of primitive elements.

We say that A is *atomic*—or is an *atom*—when $P\cap X$ contains exactly one element. It is straightforward to check that this is independent of the choice of cross-section. (This clever definition is due to Zelevinski.)

First here is a result which motivates the tensor product decomposition theorem, and gives a strong uniqueness to these decompositions.

Proposition 4.1. *In any tensor product of atomic L-PSH-algebras, the only sub-L-PSH-algebras which are atoms are the factors (in the tensor product) themselves .*

Here we are identifying $\{\ a\otimes 1\ :\ a\in A\ \}\subset A\otimes B$ with A itself, and similarly for the right-hand factor and for tensor products of more than two atoms, even infinitely many.

Proof. The proof for more than two factors is only more complicated in notation, so we'll give it for two. Let A and B be atoms, with $X_A \cap P_A = \{y\}$ and $X_B \cap P_B = \{z\}$. We take $X_{A\otimes B}$ to be $\{ u \otimes v \ : \ u \in X_A, \ v \in X_B \}$. The primitives in $A \otimes B$ are

$$P_{A\otimes B} \ = \ \{q \otimes 1 \ : \ q \in P_A \} \ \cup \ \{1 \otimes r \ : \ r \in P_B \} .$$

Now

two atomic subalgebras cannot have the same irreducible primitive element $(*)$

(proof below), so it remains to show that the elements $y \otimes 1$ and $1 \otimes z$ are the only ones in $X_{A\otimes B} \cap P_{A\otimes B}$. (Recall that $1 \in X$ is assumed.)

But $q \otimes 1 = u \otimes v$ implies that $v \in B_0$. Then $v \in X_B$ implies that $v = 1$. Now $q \otimes 1 = u \otimes 1$ implies that $q = u$, by freeness over L. But $q = u \in X_A \cap P_A$ implies $q = y$.

Similarly $1 \otimes r = u \otimes v$ with $r \in P_B$, $u \in X_A$ and $v \in X_B$ implies that $1 \otimes r = 1 \otimes z$.

There ought to be a simpler proof of $(*)$, but it certainly follows from the proof of **4.2** below: if p is the irreducible primitive in question, then those atomic subalgebras would each coincide with $A_{(p)}$ defined below, and therefore with each other.

Theorem 4.2. *Any L-PSH-algebra is isomorphic, using the multiplication map, to the tensor product of all its atomic sub-L-PSH-algebras* (which we'll call its *atoms*).

This is the main result here; the proof below is divided into many small steps. By **4.1**,
(1) these tensor factors are unique, and
(2) we need only show that A has *some* atomic sub-L-PSH-algebras into which it decomposes.

The present writeup of the proof of **4.1** makes it also depend on part of the proof of **4.2**, but it is easy to see that there is no circularity. This complication seemed worth suffering in order to have **4.1** and **4.2** stated separately and succinctly.

Here is the notation which will be used in the proof. Fix the L-PSH-algebra A to be decomposed, and a cross-section X for it. Let P be the set of primitives. For each $p \in P \cap X$, let

$$I_{(p)} \ := \ \{ \ y \in I \ : < y \ , \ p^n > \neq 0 \ \text{ for some } n \geq 0 \ \} .$$

Then $I_{(p)}$ is closed under the action of G, and has $X_{(p)} = X \cap I_{(p)}$ as a cross-section. Let

$$A_{(p)} \ := \ \mathrm{Span}_L I_{(p)} \ = \ \mathrm{Span}_L X_{(p)} \ .$$

Choose some fixed linear order for $P \cap X$. We'll consider functions $\omega : P \cap X \to \mathbf{N}$ which have *finite support*, that is, are zero on all but finitely many elements. Define

$$\pi_\omega \ := \prod_{p \in P\cap X} p^{\omega(p)} \qquad \textit{(multiplied in the given order)}.$$

If the linear order is changed, then π_ω is altered only by multiplication by an invertible factor $\rho(g) \in L_0$ for some $g \in G$. Define

$$X_\omega \ := \ \{ \ x \in X \ : < x, \ \pi_\omega > \neq 0 \ \} \ ,$$

and

$$A_\omega := \mathrm{Span}_L X_\omega .$$

The proof of **4.2** consists of **(I)** to **(X)** below.

(I) *Let Y be any orthogonal subset of P (for example, $Y = P \cap X$). If $(y_1, \cdots, y_r)$ and $(y'_1, \cdots, y'_s)$ are both sequences from Y and $< y_1 \cdots y_r \, , \, y'_1 \cdots y'_s > \neq 0$, then $r = s$ and the sequence $(y'_1, \cdots, y'_s)$ is a rearrangement of $(y_1, \cdots, y_r)$.*

Proof. Proceed by induction on r.
For $r = 1$, the case $s = 1$ is clear. But $s > 1$ cannot occur, since

$$< y_1 \, , \, y'_1 \cdots y'_s > = < y_1 \otimes 1 + 1 \otimes y_1 \, , \, y'_1 \otimes (y'_2, \cdots y'_s) >$$

$$= < y_1 \, , \, y'_1 >< 1 \, , y'_2, \cdots y'_s > + < 1 \, , \, y'_1 >< y_1 \, , \, y'_2, \cdots y'_s >$$

$$= < y_1 \, , \, y'_1 > 0 + 0 < y_1 \, , \, y'_2, \cdots y'_s > = 0 .$$

For the inductive step,

$$< y_1 \cdots y_r \, , \, y'_1 \cdots y'_s > = < \Delta(y_1 \cdots y_r) \, , \, y'_1 \otimes (y'_2 \cdots y'_s) >$$

$$= < \prod_{1 \le i \le r} (y_i \otimes 1 + 1 \otimes y_i) \, , \, y'_1 \otimes (y'_2 \cdots y'_s) >$$

$$= \sum \rho_I < y_{i_1} \cdots y_{i_k} \, , \, y'_1 >< y_{j_1} \cdots y_{j_{r-k}} \, , \, y'_2 \cdots y'_s > ,$$

where the summation is over all subsequences $I = (i_1, \cdots, i_k)$ of $(1, \cdots, r)$, with $(j_1, \cdots, j_{r-k})$ being the complementary subsequence, and with ρ_I being an invertible in L whose value is irrelevant. By the case $r = 1$, all the summands are zero except those where $k = 1$ and $y_{i_1} = y'_1$. And by the inductive hypothesis, all these summands are zero unless $r = s$ and $(y_1, \cdots \hat{y}_{i_1} \cdots, y_r)$ is a rearrangement of $(y'_2, \cdots, y'_s)$

(II) The set X *is the disjoint union of all the X_ω; and $X_{(p)}$ is the disjoint union of all the X_ω for which $\omega(y) = 0$ for all $y \neq p$ (that is, those ω whose support is $\{p\}$ or is empty.)*

Proof. To prove disjointness of the sets X_ω, suppose that $x_1 \in X$ and that

$$< x_1 \, , \, \pi_\omega > \neq 0 \neq < x_1 \, , \, \pi_{\omega'} > .$$

Now both π_ω and $\pi_{\omega'}$ are positive linear combinations from X, and the coefficient of x_1 in both cases is strictly positive by the display above. But then $< \pi_\omega \, , \, \pi_{\omega'} >$ is the sum of all the products of these coefficients, one product for each element of X. Thus it is a sum of positive elements of L, at least one of which is non-zero. Thus $< \pi_\omega \, , \, \pi_{\omega'} > \neq 0$, and so $\omega = \omega'$, as required, by **(I)**.

To show that $X \subset \bigcup_\omega X_\omega$, proceed by induction on the **N**-degree of $y \in X$. If that degree is zero, then $y = 1 = \pi_\omega$ for that ω which is 0 on everything. If y is primitive, we have $y = \pi_\omega$ for that ω which is 1 on y and 0 on everything else. Otherwise, we can write

$$\Delta(y) \;=\; y \otimes 1 \;+\; 1 \otimes y \;+\; \sum_{x,x' \in X} \mu_{x,x'} x \otimes x' \quad ,$$

where there is at least one non-zero term, say, $\mu_{x_0,x_0'} x_0 \otimes x_0'$ in the summation, where the coefficients μ are positive in L, and where the **N**-degrees of both x and x' are less than that of y. By the inductive hypothesis, we have $x_0 \in X_\omega$ and $x_0' \in X_{\omega'}$ for some ω and ω'. We'll show that y is in $X_{\omega+\omega'}$ by checking that $< y \, , \, \pi_{\omega+\omega'} >$ is non-zero. But $\pi_{\omega+\omega'}$ differs from $\pi_\omega \pi_{\omega'}$ only by an invertible in L, and

$$< y \, , \, \pi_\omega \pi_{\omega'} \; > = < \Delta(y) \, , \, \pi_\omega \otimes \pi_{\omega'} \; > = \sum_{x,x' \in X} \mu_{x,x'} < x \otimes x' \, , \, \pi_\omega \otimes \pi_{\omega'} \; > \quad .$$

The latter is a sum of positives in L, one of which, namely $\mu_{x_0,x_0'} < x_0 \, , \, \pi_\omega >< x_0' \, , \, \pi_{\omega'} >$, is non-zero, as required.

Finally, the definition of $X_{(p)}$ shows that it is the union of those X_ω for which $\omega(z) = 0$ for all $z \neq p$.

(III) *A is the direct sum of all the* A_ω *; and* $A_{(p)}$ *of those with* support$(\omega) \subset \{p\}$.

Proof. Since each of A, A_ω and $A_{(p)}$ respectively is the free L-module on X, X_ω and $X_{(p)}$, this follows immediately from **(II)**.

(IV) $A_\omega \cdot A_{\omega'} \subset A_{\omega+\omega'}$ *and* $\Delta(A_{\omega''}) \subset \bigoplus_{\omega+\omega'=\omega''} A_\omega \otimes A_{\omega'}$.

Proof. The second statement follows by self-duality of A from the first, and the first is deduced as follows. It suffices to show that $xx' \in A_{\omega+\omega'}$ for $x \in X_\omega$ and $x' \in X_{\omega'}$. We must show that, for $y \in X$, we have

$$< xx' \, , \, y \; > \neq 0 \quad \Rightarrow \quad < y \, , \, \pi_{\omega+\omega'} \; > \neq 0 \; .$$

By adjointness of multiplication and comultiplication, this is the same as

$$< x \otimes x' \, , \, \Delta(y) \; > \neq 0 \quad \Rightarrow \quad < \Delta(y) \, , \, \pi_\omega \otimes \pi_{\omega'} \; > \neq 0 \quad ,$$

since $\pi_\omega \pi_{\omega'}$ is the same as $\pi_{\omega+\omega'}$ up to an invertible in L_0. This last displayed implication follows since $\Delta(y)$ is a positive linear combination of elements in X tensored together, and the the left side of the display immediately above says that the coefficient of $x \otimes x'$ is non-zero. Thus the right side is a sum of positives in L, at least one of which is non-zero, since $< x \, , \, \pi_\omega > \neq 0 \neq < x' \, , \, \pi_{\omega'} >$.

(V) *If* $x, \; y \in X_\omega$, *and* $x', \; y' \in X_{\omega'}$, *where* ω *and* ω' *have disjoint support, then*

$$< xx', \; yy' \; > = < x, \; y >< x', \; y' > = \; \delta_{x,y} \cdot \delta_{x',y'} \qquad \text{(Kronecker deltas)} \; .$$

Proof. With 'coefficients' α snd β being positive elements of L, write down the unique expressions

$$\Delta(x) = \sum_{z,w\in X} \alpha_{z,w} z \otimes w$$

and

$$\Delta(x') = \sum_{u,v\in X} \beta_{u,v} u \otimes v \ .$$

Then, using the fact that Δ is a ring morphism, as well as adjointness of multiplication and comultiplication, we find that

$$< xx' \, , \, yy' > = < \Delta(xx') \, , \, y\otimes y' > = \sum_{X^4} \alpha_{z,w}\beta_{u,v}\nu(u,w) < zu \, , \, y >< wv \, , \, y' > \qquad (*)$$

Note that the right-hand side of $(*)$ has one summand just what we want, namely $< x, \ y >< x', \ y' >$, by using the terms $x \otimes 1$ and $1 \otimes x'$ in $\Delta(x)$ and $\Delta(x')$, respectively.

It remains to show that no other summands on the right-hand side of $(*)$ are non-zero. Consider any non-zero summand. We have, using the second half of **(IV)**,

$$\alpha_{z,w} \neq 0 \quad \Rightarrow \quad z \in X_{\omega_1} \text{ and } w \in X_{\omega_2} \quad \text{with} \quad \omega_1 + \omega_2 = \omega \ ,$$

and

$$\beta_{u,v} \neq 0 \quad \Rightarrow \quad u \in X_{\omega_3} \text{ and } v \in X_{\omega_4} \quad \text{with} \quad \omega_3 + \omega_4 = \omega' \ .$$

Furthermore, comparing subscripts on X and using the first part of **(II)**,

$$< zu \, , \, y > \neq 0 \quad \Rightarrow \quad \omega_1 + \omega_3 = \omega \ ,$$

and

$$< wv \, , \, y' > \neq 0 \quad \Rightarrow \quad \omega_2 + \omega_4 = \omega' \ .$$

The four 'ω-equations' imply that $\omega_2 = \omega_3$, and they are in fact 0 because of disjointness of the supports of ω and ω'. It now follows from the general properties of the coproduct Δ that there is only the one possibly non-zero term on the right in $(*)$, namely that with

$$w = u = 1, \quad z = x, \quad v = x', \quad \alpha_{z,w} = \beta_{u,v} = 1 \ ,$$

as required.

(VI) *For all $x \in X_\omega$, and $x' \in X_{\omega'}$, where ω and ω' have disjoint support, there is an invertible $\ell_{x,x'} \in L$ such that*

$$X_{\omega+\omega'} = \{ \ \ell_{x,x'} xx' \ : \ x \in X_\omega \ , \ x' \in X_{\omega'} \ \} \ .$$

Proof. Using **(V)**, we see that $< xx' \, , \, xx' > = 1$. But xx' is positive by P, so $\ell xx' \in X$ for some invertible $\ell \in L$. Actually it is in $X_{\omega+\omega'}$ by disjointness of the sets

X_ω, and by the first half of **(IV)**. Since π_ω is a linear combination from X_ω, and $\pi_{\omega+\omega'}$ is a multiple by some $\rho(g_0)$ of $\pi_\omega \pi_{\omega'}$, it follows that $\pi_{\omega+\omega'}$ is a linear combination of the right-hand side in the statement, so the products $\ell x x' \in X$ above exhaust all of $X_{\omega+\omega'}$, as required.

(VII) *If $x,\ y \in X_\omega$, and $x',\ y' \in X_{\omega'}$, where ω and ω' have disjoint support, and if $yy' = \ell x x'$ for some invertible $\ell \in L$, then $x = y$ and $x' = y'$.*

Proof. Using **(V)**,

$$0 \neq < \ell x x' \ , \ yy' \ > = \ \ell < x \ , \ y \ >< \ x' \ , \ y' \ > \ .$$

Thus $< x \ , \ y \ > \neq \ 0 \ \neq < \ x' \ , \ y' \ >$. Being in X, we conclude that $x = y$ and $x' = y'$.

(VIII) *If ω and ω' have disjoint support, then the multiplication map,*

$$\mu \ : \ A_\omega \otimes_L A_{\omega'} \to A_{\omega+\omega'} \ ,$$

is an isomorphism.

Proof. By definition of A_ω, it is clear that X_ω is an L-basis, since the latter is linearly independent, being a subset of X. Thus $\{ \ x \otimes x' \ : \ x \in X_\omega \ , \ x' \in X_{\omega'} \ \}$ is a basis for the domain in the statement, and it remains only to show that $\{ \ xx' \ : \ x \in X_\omega \ , \ x' \in X_{\omega'} \ \}$ is a basis for the codomain (more precisely, an indexed basis, that is, we use also that distinct pairs (x, x') never give the same product xx' nor the same $x \otimes x'$). This is immediate from **(VI)** and **(VII)**, since $X_{\omega+\omega'}$ is certainly a basis for $A_{\omega+\omega'}$.

(IX) *For any $p \in P \cap X$, the tuple $[A_{(p)},\ \mu|_{A_{(p)} \otimes A_{(p)}},\ \Delta|_{A_{(p)}},\ I_{(p)}]$ is an atomic sub-L-PSH-algebra of A.*

Proof. It is immediate from **(III)** and **(IV)** that $A_{(p)}$ is a subobject of A both regarded as an algebra and as a coalgebra. By their definitions, both $I_{(p)}$ and $X_{(p)}$ span $A_{(p)}$ over L, and $X_{(p)}$ is linearly independent, being a subset of X. The inner product obtained from $X_{(p)}$ is just the restriction to $A_{(p)} \times A_{(p)}$ of the inner product on A. Axioms P and S are then immediate. Thus we have a sub-L-PSH- algebra. It is atomic since $P \cap X_{(p)}$ contains only p, and since the primitives in $A_{(p)}$ are necessarily primitive in A.

(X) *Multiplication defines an isomorphism*

$$\bigotimes_{p \in P \cap X} A_{(p)} \to A$$

of L-PSH-algebras.

Proof. This map is surjective because A is spanned by the A_ω [by **(III)**], and , for distinct p_i's, the part $A_{(p_1)} \otimes A_{(p_2)} \otimes \cdots \otimes A_{(p_r)}$ of the domain maps onto the direct sum of all the A_ω for those ω which have support $\{p_1, \cdots, p_r\}$ [by iterating **(VIII)**].

To check injectivity, consider an element in the kernel. It is in $A_{(p_1)} \otimes A_{(p_2)} \otimes \cdots \otimes A_{(p_r)}$ for some $\{p_1, \cdots, p_r\}$, so maps to zero in the the direct sum of the A_ω immediately above. Therefore it is zero by **(VIII)**.

That multiplication preserves the L-PSH-algebra structure is immediate.

By **(IX)** and **(X)**, the proof of **4.2** is complete.

Classification of atoms for a given base ring L will depend on it, so we record the following.

Corollary to the proof 4.3. *If A is an atom with irreducible primitive p and special irreducible set I, then every $a \in I$ satisfies $< a \ , \ p^n \ > \neq \ 0$ for some $n \geq 0$.*

Proof. We have $A = A_{(p)}$ and $I = I_{(p)}$, with notation from the proof of **4.2**, so this follows from the definition of $I_{(p)}$.

5. Examples of L-PSH-algebras.

The original example of Zelevinsky **[Z]** arises by taking the group G to be zero, and letting $L = \mathbf{Z}$. So $\mathcal{B}$ must be $\{1\}$, and the conditions are satisfied. A major result in **[Z]** is the classification of atoms. See **[H-T]** for a fairly short proof. As ungraded rings, the atoms here are all isomorphic to the polynomial ring, $\mathbf{Z}[x_1, \ x_2, \ \cdots]$, on countably many variables. The bottom variable x_1 can have any positive grading d, and then x_n is in grading nd. We have (with $x_0 := 1$)

$$\Delta(x_n) \ = \ \sum_{i+j=n} x_i \otimes x_j \ .$$

Finally, positivity in the atom is not so easy to define. Under a certain standard isomorphism, due to Frobenius, of this algebra with the direct sum of the Grothedieck groups of linear representations of the symmetric groups, the set I corresponds to the irreducible representations. Alternatively, I consists of those polynomials in the x_i which will produce Jacobi's Schur functions when the elementary symmetric functions are substituted for the variables x_i.

The example which relates to projective representations of the symmetric group has $G = \mathbf{Z}/2$, and $L = \mathbf{Z}[\lambda]/ < \lambda^3 - 2\lambda >$ as an ungraded ring. One puts λ in $\mathbf{Z}/2$-grading 1, so that

$$L_0 \cong \mathbf{Z}^2 \quad \text{with basis} \quad \mathcal{B}_0 \ = \ \{1, \ \theta\} \quad \text{where} \quad \lambda^2 = 1 + \theta \ ,$$

and

$$L_1 \cong \mathbf{Z} \quad \text{with basis} \quad \mathcal{B}_1 \ = \ \{\lambda\} \ .$$

The map ρ is the isomorphism from $\mathbf{Z}/2$ to $\{1, \ \theta\}$. The twist map ingredient, ϵ, is defined by $\epsilon[(\alpha, i) \ , \ (\beta, j)] \ = \ \alpha\beta + \overline{ij}$, where 'barring' is reduction $(mod\, 2)$. In this case there are four distinct atoms, up to the operation of re-grading by multiplying all $\mathbf{N}$-gradings by a constant. See **[B-H]**. The more interesting two of these four are closely related to Schur's

Q-functions. See **[H-H]** for applications to representations of several other families of groups.

There is a way of generating new examples from old by taking Cartesian products of the groups G combined with tensor products of the rings L. This is explained in the next paragraph. If the ring in the preceding paragraph is denoted $L^{(1)}$, we can then take $G = (\mathbf{Z}/2)^r$ and $L^{(r)} = (L^{(1)})^{\otimes r}$ to get the sequence promised earlier; the previous two examples are when $r = 0$ and 1. This sequence of examples will apply to the representation theory of certain sequences of covering groups of monomial groups, where a total of "r" distinct 'sign homomorphisms' on each of these groups is taken as part of the structure. For $r \geq 2$, the classification of atoms given in the next section is new. It was done for the first two cases in **[Z]** and **[B-H]**, as noted above. The proof of decomposition as a tensor product for this sequence of examples was done in **[W]**.

To see how to generate these examples by tensor products, let L_1 and L_2 be two examples, corresponding to abelian groups G_1 and G_2, and with positive bases $\mathcal{B}_1$ and $\mathcal{B}_2$, and twist maps defined using ρ_1 , ρ_2 and ϵ_1 , ϵ_2. Take $G = G_1 \times G_2$ and $L = L_1 \otimes_{\mathbf{Z}} L_2$. For grading, we set $L_{(g_1,g_2)} = (L_1)_{g_1} \otimes_{\mathbf{Z}} (L_2)_{g_2}$ Define

$$\mathcal{B} = \{b_1 \otimes b_2 \ : \ b_i \in \mathcal{B}_i \ \} \ .$$

Let $\rho(g_1, \ g_2) = \rho_1(g_1) \otimes \rho_2(g_2)$. Finally, define

$$\epsilon \ : \ (G_1 \times G_2 \times \mathbf{N}) \times (G_1 \times G_2 \times \mathbf{N}) \quad \longrightarrow \quad G_1 \times G_2$$

by

$$[(x, y, a), (u, v, b)] \quad \mapsto \quad [\ \epsilon_1(x, a, u, b) \ , \ \epsilon_2(y, a, v, b) \ ] \ .$$

It is straightforward to check that a product of elements in $\mathcal{B}$ is strictly positive, using that property of the $\mathcal{B}_i$, and the fact that $\ell \otimes m \neq 0$ for non-zero ℓ and m in a pair of free abelian groups.

As a final example, in general when $G = 0$, the base ring L is ungraded, and Zelevinsky's theorem that there is only one atom (up to simple regrading) is true in this generality. In fact, if A is the atom for $L = \mathbf{Z}$, then, for general ungraded L, the atom is

$$L \otimes_{\mathbf{Z}} A \ \cong \ L \otimes_{\mathbf{Z}} \mathbf{Z}[x_1, x_2, \cdots] \ \cong \ L[x_1, x_2, \cdots] \ .$$

The arguments in **[H-T]** are readily modified to prove this. As a class of examples, we can take $L = R(\Gamma)$, the representation ring of a finite group Γ. The atom is

$$\bigoplus_{k=0}^{\infty} R(\Gamma \times S_k) \ \cong \ \bigoplus_{k=0}^{\infty} [R(\Gamma) \otimes_{\mathbf{Z}} R(S_k)] \ \cong \ R(\Gamma) \otimes_{\mathbf{Z}} \bigoplus_{k=0}^{\infty} R(S_k) \ = \ L \otimes_{\mathbf{Z}} A \ .$$

The module action of $R(\Gamma)$ on $R(\Gamma \times S_k) \cong R(\Gamma) \otimes R(S_k)$ is the natural one which uses the internal tensor product of Γ-modules (i.e. the ring multiplication in $R(\Gamma)$). The old problem of describing explicitly in terms of irreducibles the internal tensor product in $R(S_\ell)$ might be worth considering from this point of view.

6. Classification of atoms for base ring $L^{(r)}$

Here we classify the atoms for all the base rings $L^{(r)}$ from the previous section, and thereby determine all the $L^{(r)}$-PSH-algebras. It seems a bit surprising that all these atoms can, in a sense, be built (and differ very little) from those which arise when $r = 1$.

Recall that, as a **Z**-module, $L^{(1)}$ is free of rank 3, with positive basis $\{1, \theta, \lambda\}$. So $L^{(r)} = (L^{(1)})^{\otimes r}$ is free of rank 3^r. Taking $\theta_j := 1\otimes\cdots\otimes\theta\otimes\cdots\otimes 1$ and $\lambda_j := 1\otimes\cdots\otimes\lambda\otimes\cdots\otimes 1$ (all 1's except in the jth slot), the positive basis for $L^{(r)}$ is $\{\theta_\alpha\lambda_\beta\}$, as (α, β) ranges over disjoint pairs of subsets of $\{1, \cdots, r\}$, and where θ_α is the product of the θ_j for j in α, and similarly for λ_β.

Theorem 6.1 *If K is an atomic $L^{(r)}$-PSH-algebra with irreducible primitive k_1 in grading (a, b), where $a = (g_1, g_2, \ldots, g_r) \in (\mathbf{Z}/2)^{\oplus r}$, then $g_j \not\equiv b (\text{mod } 2)$ for at most one value of j.*

(a) If $g_j \equiv b (\text{mod } 2)$ for all j, then there are irreducibles $k_1, k_2 \ldots$ such that

i) $K \cong L^{(r)}[k_1, k_2, \ldots]$, the polynomial algebra over $L^{(r)}$;

ii) $\Delta(k_n) = \sum_{s=0}^{n} k_s \otimes k_{n-s}$ (with $k_0 := 1$);

iii) $k_n \in K_{(an,bn)}$.

(b) If $g_j \equiv b (\text{mod } 2)$ for all $j \neq i$, with g_i of opposite parity, there are irreducibles $k_1, k_2, \ldots$ such that

i) $K \cong$ (free associative $L^{(r)}$ algebra on $\{k_1, k_2, \ldots\})/I$ *as an $L^{(r)}$-algebra, where I is the ideal generated by the following relations (called the "squaring" and "pseudo-commutativity" relations) :*

$$k_n^2 - (-1)^{n+1}\lambda_i \left[k_{2n} + \lambda_i \sum_{s=1}^{n-1} (-1)^s k_s k_{2n-s}\right] ;$$

$$k_n k_m - \theta_i^\ell k_m k_n ;$$

with $\ell = 1$ if b is even, and $\ell = m + n + 1$ if b is odd.

ii) $\Delta(k_n) = k_n \otimes 1 + 1 \otimes k_n + \lambda_i \sum_{s=1}^{n-1} k_s \otimes k_{n-s}$.

iii) $k_n \in K_{(a+b(n+1,n+1,\ldots,n+1),bn)}$.

The grading restriction on k_1, proved next, is the crucial result which rules out "new" atoms for $r \geq 2$.

Lemma 6.2 *If k_1, as in* **6.1**, *has grading (a, b), where $a = (g_1, g_2, \ldots, g_r) \in (\mathbf{Z}/2)^{\oplus r}$, then $g_j \not\equiv b (\text{mod } 2)$ for at most one value of j, for any $r \geq 1$.*

Proof. First, we calculate as follows, recalling that the coproduct is a morphism of rings adjoint to multiplication, using the twist map when multiplying in $K \otimes K$.

$$< k_1^2, k_1^2 > = < k_1 \otimes k_1, \Delta(k_1^2) > = < k_1 \otimes k_1, (\Delta k_1)^2 >$$
$$= < k_1 \otimes k_1, (1 \otimes k_1)^2 + (k_1 \otimes 1)(1 \otimes k_1) + (1 \otimes k_1)(k_1 \otimes 1) + (k_1 \otimes 1)^2 >$$
$$= < k_1 \otimes k_1, (k_1 \otimes 1)(1 \otimes k_1) + (1 \otimes k_1)(k_1 \otimes 1) >$$
$$= < k_1 \otimes k_1, k_1 \otimes k_1 + \nu(k_1, k_1) k_1 \otimes k_1 > = 1 + \nu(k_1, k_1) .$$

The formula (where Δ is the symmetric difference),

$$(\theta_{\alpha_1}\lambda_{\beta_1})(\theta_{\alpha_2}\lambda_{\beta_2}) = \sum_{\gamma \subset \beta_1 \cap \beta_2} \theta_{(\alpha_1 \Delta \alpha_2)\Delta\gamma} \lambda_{\beta_1 \Delta \beta_2} , \qquad \text{(I)}$$

is readily proved by first doing the cases only involving θ, and only λ. The coefficient of 1 in this formula is zero if either $\alpha_1 \neq \alpha_2$ or $\beta_1 \neq \beta_2$. A special case of (I) is

$$(\theta_\alpha \lambda_\beta)^2 = 1 + \sum_{\emptyset \neq \gamma \subset \beta} \theta_\gamma . \qquad \text{(II)}$$

If k_1 is in grading $((g_1, g_2, \ldots, g_r), b)$, then $\nu(k_1, k_1) = \prod_{i=1}^r \theta_i^{g_i^2 + b^2}$. Assume, for a contradiction, that $g_i^2 + b^2$ is odd for more than one i. We shall show that $y = 1 + \nu(k_1, k_1)$ is not a sum of squares of positive elements of $L^{(r)}$, giving the contradiction, since the inner product of a positive element, k_1^2, with itself must be such a sum.

By (II) and the remark immediately after (I), the square, $(\sum n_{\alpha,\beta}\theta_\alpha\lambda_\beta)^2$, of a positive element has $\sum n_{\alpha,\beta}^2$ as the coefficient of 1 when written as a linear combination of the positive basis. Now y is $1 + \theta_\delta$ for some δ with at least two elements. For it to be a sum of squares of positive elements $\sum n_{\alpha,\beta}\theta_\alpha\lambda_\beta$ as above, the 'leading term' 1 would then be the sum of the various summations $\sum n_{\alpha,\beta}^2$. So there can be at most one such $\sum n_{\alpha,\beta}\theta_\alpha\lambda_\beta$, and it must simply be $\theta_\alpha\lambda_\beta$. But (II) shows that $(\theta_\alpha\lambda_\beta)^2 \neq 1 + \theta_\delta$ for any δ with at least two elements, giving the required contradiction.

Without loss of generality, assume for the rest of the proof that $i = 1$ in (b) of **6.1**.

Lemma 6.3 *There is a sequence of irreducible elements $k_1, k_2, \ldots$ in K such that*

$$k_1 k_{n-1} = \begin{cases} k_n + u_n & g_1, \ldots g_r, b \text{ of same parity ;} \\ \lambda_1 k_n + u_n & g_1 \text{ of different parity ;} \end{cases}$$

where u_n is an irreducible element unless $n = 2$ and g_1 is of different parity, in which case $u_2 = 0$. Furthermore, in each case, both $< k_1^s, k_n >$ and $< k_1^s, k_1^s >$ are in "$L = L^{(1)} \otimes 1 \otimes \cdots \otimes 1$" $\subset L^{(r)}$.

N.B. For the remainder of the paper, we appropriate L for the name of this subring, isomorphic to $L^{(1)}$, of $L^{(r)}$ (rather than using it for the general base ring, as in the previous sections).

Proof. We pause first to derive the handy formula

$$k_1^\perp(ab) = \nu(a, k_1) a k_1^\perp(b) + k_1^\perp(a) b ,$$

where $x^\perp$ is defined by

$$< x^\perp(y)\ ,\ z\ > = < y\ ,\ xz\ > \ .$$

$$< k_1^\perp(ab), u\ > = < ab, k_1 u\ > = < a\otimes b, \Delta k_1 \Delta u\ >$$
$$= < a\otimes b, (1\otimes k_1)\Delta u\ > + < a\otimes b, (k_1\otimes 1)\Delta u\ >$$
$$= < a\otimes b, \nu(a,k_1)\Delta u(1\otimes k_1)\ > + < a\otimes b, (k_1\otimes 1)\Delta u\ >$$
$$= \nu(a,k_1) < a\otimes k_1^\perp(b), \Delta u\ > + < k_1^\perp(a)\otimes b, \Delta u\ > = < \nu(a,k_1)ak_1^\perp(b) + k_1^\perp(a)b, u\ > \ .$$

Now to begin the proof in earnest, first assume we are in case (a) of **6.1**, that is, $g_j \equiv b(\text{mod } 2)$ for all j. We shall prove the stronger statement that the k_n desired exist and satisfy $k_1^\perp(k_n) = k_{n-1}$. As discussed in **6.2**, $< k_1^2, k_1^2\ >= 2$.

By an appropriate choice of cross-section, we may assume that $k_1^2 = k_2 + y_2$ for some irreducibles k_2 and y_2. For the inductive step, let us assume that the formulae for $k_1 k_{n-1}$ and $k_1^\perp(k_n)$ hold.

Note that

$$< k_1 k_n, k_1 k_n\ > = < k_1\otimes k_n, \Delta k_1 \Delta k_n\ >$$
$$= 1 + < k_1\otimes k_n, (1\otimes k_1)(k_1\otimes k_1^\perp(k_n))\ > = 1 + < k_n, k_1 k_{n-1}\ > = 2\ .$$

Thus there are irreducibles x_1, x_2 such that $k_1 k_n = x_1 + x_2$ (again up to choice of cross-section). Furthermore, using the formula for $k_1^\perp(ab)$, this gives

$$k_1^\perp(x_1) + k_1^\perp(x_2) = k_1^\perp(x_1 + x_2) = k_1^\perp(k_1 k_n) = k_n + k_1 k_{n-1} = 2k_n + u_n\ .$$

But, $< k_1^\perp(x_1), k_n\ >=< x_1, k_1 k_n\ >= 1$ and similarly $< k_1^\perp(x_2), k_n\ >= 1$. So we must have that one of the two terms $k_1^\perp(x_1)$ and $k_1^\perp(x_2)$ is $k_n + u_n$ and the other k_n. Let the appropriate one be called k_{n+1} and the other u_{n+1} so that $k_1^\perp(k_{n+1}) = k_n$, and the induction is complete.

Now assume we are in case (b) of **6.1**, that is, $g_j \equiv b(\text{mod } 2)$ for all $j > 1$, but g_1 has the opposite parity. This gives $< k_1^2, k_1^2\ >= 1 + \theta_1$. By an appropriate choice of cross-section, we may assume that $k_1^2 = \lambda_1 k_2$ for some irreducible k_2. Then

$$\Delta k_2 = 1\otimes k_2 + \lambda_1 k_1 \otimes k_1 + k_2\otimes 1\ .$$

Furthermore

$$< k_1 k_2, k_1 k_2\ > = < k_1\otimes k_2, (\Delta k_1)(\Delta k_2)\ >$$
$$= < k_1\otimes k_2, (1\otimes k_1)(\lambda_1 k_1\otimes k_1) + (k_1\otimes 1)(1\otimes k_2)\ >= 1 + \nu(k_1,k_1)\lambda_1 < k_2, k_1^2\ > = 2 + \theta_1\ .$$

Thus for some cross-section, we have $k_1 k_2 = \lambda_1 k_3 + u_3$ (that there are no more possibilities follows from an argument similar to that given in **6.2**). Noting that k_2 being the only irreducible in **N**-grading 2 gives $k_1^\perp(k_3) = \lambda_1 k_2$, a similar calculation reveals irreducibles k_4 and u_4 such that $k_1 k_3 = \lambda_1 k_4 + u_4$.

Again, we use induction to establish the stronger statement that the k_n exist and satisfy $k_1^\perp(k_n) = \lambda_1 k_{n-1}$. Let us assume that the formulae for $k_1 k_{n-1}$ and $k_1^\perp(k_n)$ hold.

Note that

$$< k_1k_n, k_1k_n > = < k_1 \otimes k_n, \Delta k_1 \Delta k_n > = 1 + < k_1 \otimes k_n, (1 \otimes k_1)(k_1 \otimes k_1^{\perp}(k_n)) >$$

$$= 1 + \nu(k_1, k_1) < k_n, \lambda_1 k_1 k_{n-1} > = 2 + \theta_1 \ .$$

Thus there are irreducibles k_{n+1}, u_{n+1} (again up to choice of cross-section) such that $k_1k_n = \lambda_1 k_{n+1} + u_{n+1}$. Note that $k_1^{\perp}(k_1k_n) = k_n + \lambda_1 k_1 k_{n-1}$. Using this identity,

$$\lambda_1 k_1^{\perp}(k_{n+1}) + k_1^{\perp}(u_{n+1}) = k_1^{\perp}(\lambda_1 k_{n+1} + u_{n+1}) = k_1^{\perp}(k_1k_n)$$

$$= k_n + \lambda_1 k_1 k_{n-1} = (1 + \lambda_1^2)k_n + \lambda_1 u_n \ .$$

But, $< k_1^{\perp}(k_{n+1}), k_n > = < k_{n+1}, k_1k_n > = \lambda_1$. Thus we find that there are several possible positive solutions (α, β) defined by

$$k_1^{\perp}(k_{n+1}) = \lambda_1 k_n + \beta u_n \ .$$

$$k_1^{\perp}(u_{n+1}) = k_n + \alpha u_n \ .$$

$$\alpha + \lambda_1 \beta = \lambda_1 \ .$$

To narrow the possibilities, since $n \geq 4$, we can calculate

$$< k_1^2 k_{n-1}, k_1^2 k_{n-1} > = < k_1 k_{n-1}, (1 + \theta_1) k_1 k_{n-1} + \theta_1^{\ell} \lambda_1 k_1^2 k_{n-2} >$$

$$= < k_{n-1}, (1 + \theta_1) k_{n-1} + (1 + \theta_1) \lambda_1 k_1 k_{n-2} + (1 + \theta_1) \lambda_1 k_1 k_{n-2} + \lambda_1^2 k_1^2 k_{n-2} >$$

$$= \lambda_1^2 + 2\lambda_1^4 + \lambda_1^4 = 7\lambda_1^2 \ ,$$

and

$$< k_1^2 k_{n-1}, k_1 k_n > = < k_1 k_{n-1}, k_n + \lambda_1 k_1 k_{n-1} > = \lambda_1 + \lambda_1(\lambda_1^2 + 1) = 4\lambda_1 \ .$$

If we let $k_1^2 k_{n-1} = \sigma u_{n+1} + \tau k_{n+1} + \sum_i \gamma_i u_i$ for the irreducibles u_i distinct from u_n and k_n, this yields equations

$$\sigma^2 + \tau^2 + \sum_i \gamma_i^2 = 7\lambda_1^2 \ .$$

$$\sigma + \lambda_1 \tau = 4\lambda_1 \ .$$

Considering the nonnegative possibilities gives $\tau = a\lambda_1^2$ for $a = 1$ or 2. But $a = 2$ is not compatible with the first equation. So $< k_1^2 k_{n-1}, k_{n+1} > = \lambda_1^2$. But

$$< k_1^2 k_{n-1}, k_{n+1} > = < k_1 k_{n-1}, k_1^{\perp}(k_{n+1}) > = < \lambda_1 k_n + u_n, k_1^{\perp}(k_{n+1}) > = \lambda_1^2 + \beta \ .$$

Therefore $\beta = 0$ and hence, $k_1^{\perp}(k_{n+1}) = \lambda_1 k_n$, completing the induction.

Next we prove, simultaneously for both cases above, that $< k_1^s, k_n > \in L$ for all $n > 0$ by induction on $s \geq 0$. The initial cases $s = 0$ or 1 are trivial. Note that, independently of

k_1's grading, for $n > 1$, $k_1^{\perp}(k_n) = a_n k_{n-1}$ for some $a_n \in L$, by the inductions completed above in both cases. But then, for any $n > 1$,

$$< k_1^s, k_n > = a_n < k_1^{s-1}, k_{n-1} > \in L ,$$

completing the induction.

Finally, we prove $< k_1^s, k_1^s > \in L$ by induction on $s \geq 1$. For $s = 1$, the answer is $1 \in L$. For the inductive step,

$$< k_1^{s+1}, k_1^{s+1} > = < k_1^s, \sum_{j=0}^{s} \nu(k_1^j, k_1) k_1^s > = \sum_{j=0}^{s} \nu(k_1^j, k_1) < k_1^s, k_1^s > .$$

But $\nu(k_1^j, k_1) \in L$ by examination of the gradings of k_1^j, completing the induction, and the proof of **6.3**.

Proof of 6.1. By **6.3**, $< k_1^s, k_1^s > \in L$. This implies that we may choose a cross-section X so that the coefficients in the linear combination of irreducibles giving k_1^s are in L. By **4.3**, for any irreducible w of K, there is an s such that $< k_1^s, w > \neq 0$. Therefore, since, by **6.3**, $< k_1^s, k_n >$ is in L, the k_n are in X.

Let $R = Span_L(X) \subset K$. The basic idea of the proof is, after regrading R to be graded by $\mathbf{Z}/2 \times \mathbf{N}$ (rather than $(\mathbf{Z}/2)^{\oplus r} \times \mathbf{N}$), we have an L-PSH-algebra to which the results of **[B-H]** may be applied. It will then be straightforward to 'pump up' results about R to get the required results about K.

For elements from R, then, the inner product takes values in L. As for regrading R, let

$$M = \{((p_1, p_2, \ldots, p_r), p_0) \in (\mathbf{Z}/2)^{\oplus r} \times \mathbf{N} : p_2, \ldots, p_r, p_0 \text{ have the same parity } \} .$$

Let $\gamma(w) \in (\mathbf{Z}/2)^{\oplus r} \times \mathbf{N}$ be the grading of the homogeneous element $w \in K$. Recall that $\gamma(k_1) \in M$. For any $w \in X$ and $s =$ the $\mathbf{N}$-grading of w, we have $< k_1^s , w > \in L$. Therefore, for some $q \in \{0, 1\}$,

$$\gamma(w) = \gamma(k_1^s) + ((q, 0, \ldots, 0), 0) .$$

So $\gamma(w) \in M$. It follows easily that all homogeneous elements of R have their grading in M. Regrade each such element by dropping its grading in all $\mathbf{Z}/2$ factors except the first.

By definition, R is closed under addition and scalar multiplication by scalars in L.

To show that R is closed under multiplication, we need only show closure for elements of X. Let $u_n, u_m \in X$ have $\mathbf{N}$-gradings n, m, respectively. Then $u_n u_m = \sum_{j=1}^{d} \alpha_j y_j$ ($y_j \in X$) has $\mathbf{N}$-grading $n + m$. Consider α_j for a particular j. Since $k_1^n k_1^m = k_1^{m+n}$ has coefficients in L when written as a linear combination of elements of X, $< k_1^{m+n}, y_j > \in L$. But also, $< k_1^n, u_n >= \beta_n \in L$ and $< k_1^m, u_m >= \beta_m \in L$, and therefore k_1^{m+n} is a sum of elements including $\beta_n \beta_m \alpha_j y_j$. Remembering that positivity is our saviour, we have $\beta_n \beta_m \alpha_j \in L$, and therefore $\alpha_j \in L$. Hence R is closed under multiplication.

To show that R is closed under the co-product (i.e. restricted to R, it can be regarded as taking values in $R \otimes_L R$), it is enough to show that $< \Delta w, u_1 \otimes u_2 > \in L$ for any $w \in R$ and $u_1, u_2 \in X$. But $< w, u_1 u_2 > \in L$, so this follows from closure under multiplication.

It follows that R, with the regrading specified, becomes an L-PSH-algebra. Clearly, each $k_n \in R$. And k_1 is the only primitive irreducible, so R is atomic. By **6.3**, the k_n satisfy

$$k_1 k_{n-1} = \begin{cases} k_n + u_n & g_1, b \text{ of same parity ;} \\ \lambda k_n + u_n & \text{otherwise .} \end{cases}$$

And so, by Lemma 4.3 of **[B-H]**, the following relations hold:

(i)

$$\Delta k_n = \begin{cases} \sum_{s=0}^{n} k_s \otimes k_{n-s}, & g_1, b \text{ of same parity ;} \\ k_n \otimes 1 + 1 \otimes k_n + \lambda_1 \sum_{s=1}^{n-1} k_s \otimes k_{n-s}, & \text{otherwise ;} \end{cases}$$

(ii) if g_1, b are of different parity,

$$k_n^2 = (-1)^{n+1} \lambda_1 \left[k_{2n} + \lambda_1 \sum_{s=1}^{n-1} (-1)^s k_s k_{2n-s} \right] .$$

Consequently, R is isomorphic to one of the four atoms of Theorem 2.4 of **[B-H]**, with the same k_n irreducibles. And that theorem is the same as the case $r = 1$ of **6.1**.

The proof is now quickly completed as follows. The modules R and K are both free modules with the same basis X, the former over L and the latter over $L^{(r)}$, ignoring gradings. Thus the canonical map gives an isomorphism from $L^{(r)} \otimes_L R$ to K. That map is an isomorphism of ungraded algebras and coalgebras over $L^{(r)}$. Since its domain has the structure as given in **6.1**, so does its codomain.

The only remaining question is the grading. But the gradings of the k_n are determined by the formula for $k_1 k_n$. By induction, they are as stated in **6.1**.

Remark. There is a compact operator formula which can be regarded as an explicit expression of the basis of irreducibles in terms of a basis of monomials in the generators k_i. For the PSH-algebras in (a) of **6.1**, it is exactly analogous to the operator introduced in **[Z]**, p. 69, and used in **[H-T]** to give a short proof of Zelevinsky's original theorem. It is similar but simpler than the following one, which applies to the PSH-algebras in (b) of **6.1**. Define

$$R_n \ : \ K \to K \quad \text{by} \quad R_n(x) \ := \ k_n x \ + \ \lambda_i \sum_{j>0} (-1)^j k_{n+j} k_j^{\perp}(x) \ .$$

Then a cross-section basis of special irreducibles may be chosen to be

$$\{ \ R_{n_1} R_{n_2} \ldots R_{n_\ell}(1) \ : \ n_1 > n_2 > \cdots > n_\ell > 0 \ \} .$$

The proof is closely analogous to that for the theorem in **[H]**.

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