Math 205A Final Exam (75 points)

Name: ____

- Check that you have 8 questions on four pages.
- Show all your work to receive full credit for a problem.

1. (12 points) Short answers: (Show all the calculations to get the answers. No explanations needed.)

   (a) If C is a 4×5 matrix, what is the largest possible rank of C? What is the smallest possible dimension of Nul C?

   (b) For a 3×3 matrix B, det $B = -1$. Find det $4B$.

   (c) Find the distance between the vector $\vec{u} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

   (d) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

   $$T(x_1, x_2, x_3) = (x_1 - x_2, 2x_2 - x_3).$$

   Find a matrix A such that $T(\vec{x}) = A\vec{x}$.

(e) Let $\vec{p_1}(t) = 1$, $\vec{p_2}(t) = 2t$, $\vec{p_3}(t) = 4t^2 - 2$ and $\vec{p_4}(t) = 8t^3 - 12t$. Then $\mathcal{B} = \{\vec{p_1}, \vec{p_2}, \vec{p_3}, \vec{p_4}\}$ is a basis for $\mathbb{P}_3$. Find the polynomial $\vec{q}$ in $\mathbb{P}_3$, given that $[q]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}$.

2. (4 points) Suppose the columns of a 4×4 matrix A span $\mathbb{R}^4$. Is $\det A = 0$? Explain.

3. (12 points) Determine if the following sets are subspaces of the appropriate vector spaces. If a set is a subspace, find a basis and the dimension of the subspace. If a set is not a subspace, explain why.

(a) $W = \left\{ \text{all vectors in } \mathbb{R}^3 \text{ of the form } \begin{bmatrix} r+5s \\ s-r-3t \\ 2r+5t \end{bmatrix} \text{ where r, s, t are in } \mathbb{R} \right\}.$

(b) $W = \{\text{all polynomials in } \mathbb{P}_2 \text{ of the form } \vec{\mathrm{p}}(\mathrm{t}) = \mathrm{at} + \mathrm{bt}^2, \text{where a, b are integers}\}.$

(c) $W = \{$all symmetric matrices in $\mathrm{M}_{2\times 2}\}$. (Recall that a symmetric matrix is a matrix A such that $A^T = A$.)

4. (5 points) Suppose A is a symmetric $n \times n$ matrix and B is any $n \times n$ matrix. Explain why BAB^T is orthogonally diagonalizable.

5. (12 points) Define a linear transformation $T : \mathbb{P}_1 \longrightarrow \mathbb{R}_2$ by $T(\vec{p}) = \begin{bmatrix} \vec{p}(1) \\ \vec{p}(1) \end{bmatrix}$. (Recall that a vector $\vec{p}$ in $\mathbb{P}_1$ is a polynomial of the form $a + bt$.)

(a) Find $T(3)$ and $T(2 - 7t)$.

(b) Find a polynomial p in $\mathbb{P}_1$ such that $T(\vec{p}) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ or explain why we cannot find such a polynomial.

(c) Find a polynomial that spans the kernel of T. (Recall that the kernel of T is the space of all vectors that are mapped to the zero vector under T,i.e., the kernel is the null space of T.)

(d) Is T one-to-one? Explain.

6. (12 points) Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

(a) Is 1 an eigenvalue of A? If so, find a basis for the eigenspace corresponding to 1. If not, explain why not.

(b) Is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ an eigenvector of A? If so, find the corresponding eigenvalue. If not, explain why not.

(c) Use your answers in parts (a) and (b) to diagonalize A, if possible. (That is, if possible, find matrices P and D such that $A = PDP^{-1}$.) If not, explain why not.

7. (10 points) Hurricanes develop low pressure at their centers that generates high winds. The maximum wind speed s (in knots) and the central pressure p of a hurricane are approximately related by the equation $b_0 + b_1\, p = s$. We have the following data on four recent Atlantic hurricanes in the United States.

p	905	920	960	990
s	130	110	80	60

Find b_0 and b_1 so that the model $b_0 + b_1\, p = s$ is a least-squares fit to the data. (Start by using the given data to write a system of linear equations to determine b_0 and b_1.)

8. (8 points) Let $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. Let $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$ and $\vec{y} = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$.

(a) The set $\{\vec{u}_1, \vec{u}_2\}$ is not an orthogonal set. Find two vectors in W that are orthogonal to each other and span W. (Use the vectors $\vec{u}_1$ and $\vec{u}_2$ to produce the two orthogonal vectors.)

(b) Find a vector in W that is closest to $\vec{y}$.