Toroidal Representations of Gaussian Primes

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Objective

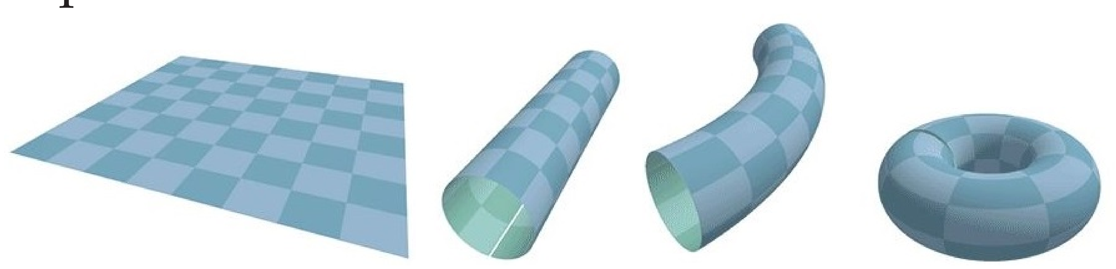
After discovering several methods of visualization for primes in $\mathbb{Z}$ using periodic waves, we searched for an analogous way to represent Gaussian primes in $\mathbb{Z}[i]$ using three-dimensional periodic functions. In order to visualize these functions more easily, we mapped each of them onto the surface of a torus. We used these patterns to examine the properties of Gaussian primes and to determine which definition of Gaussian twin primes is most analogous to the definition of twin primes in the integers.

Methods

- Gaussian integers are complex numbers of the form $a+bi$, where $a, b \in \mathbb{Z}$.
- A Gaussian prime $\alpha = a + bi$ is a Gaussian integer for which one of the following is satisfied:
  1. $\alpha = \pm 1 \pm i$,
  2. one of $\pm\alpha$ is a prime in $\mathbb{Z}$ such that $\alpha \equiv 3 \pmod 4$,
  3. The norm $N = (a+bi)(a-bi) = a^2 + b^2$ is a prime in $\mathbb{Z}$ such that $a^2 + b^2 \equiv 1 \pmod 4$.
- In order to visualize the Gaussian integers of interest, we mapped the periodic functions to the surface of a torus.
- Given a periodic function $f(x, y)$ defined on the unit square $(0,1) \times (0,1)$ in $\mathbb{R}^2$, the torus which visualizes $f(x, y)$ can be given by

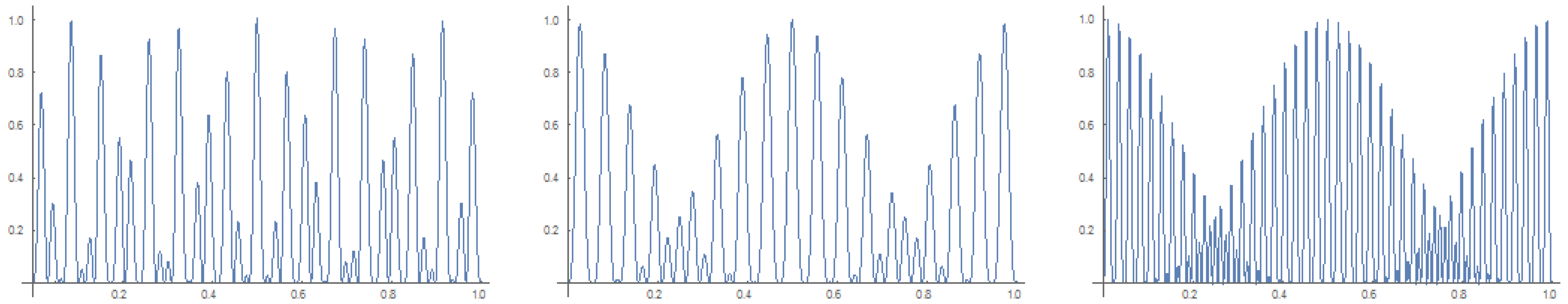
$$x(x,y) = \Big(R + \big(r + sf(x,y)\big)\cos(2\pi y)\Big)\cos(2\pi x)$$
$$y(x,y) = \Big(R + \big(r + sf(x,y)\big)\cos(2\pi y)\Big)\sin(2\pi x)$$
$$z(x,y) = \Big(r + sf(x,y)\Big)\sin(2\pi y),$$

where R is the major radius (the distance from the center of the torus to the center of the tube) and r is the minor radius (the radius of the tube). The scalar s determines the height of the peaks on the surface of the torus.



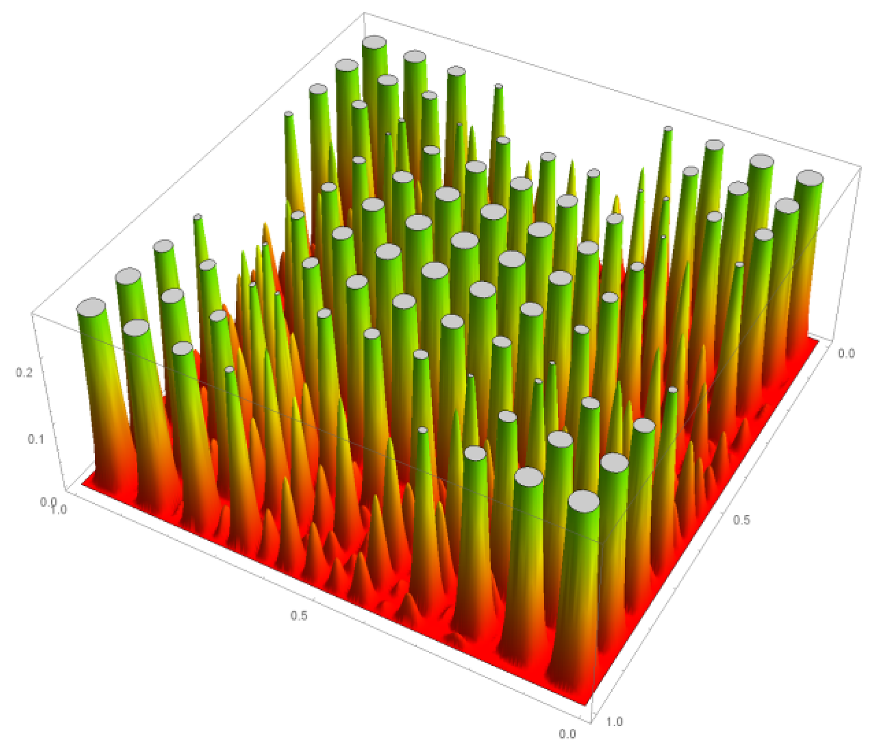
Interference Patterns

- Motivating case in $\mathbb{Z}$: In the integers, we can define a periodic function $f(x) = sin^2(p\pi x)$ that is representative of a prime p. To obtain an interference pattern between two primes, we simply multiply their corresponding functions. For example, to visualize the interference pattern created by the periodic functions representative of 41 and 43, we create a new periodic function given by $f(x) = sin^2(41\pi x) \cdot sin^2(43\pi x)$.

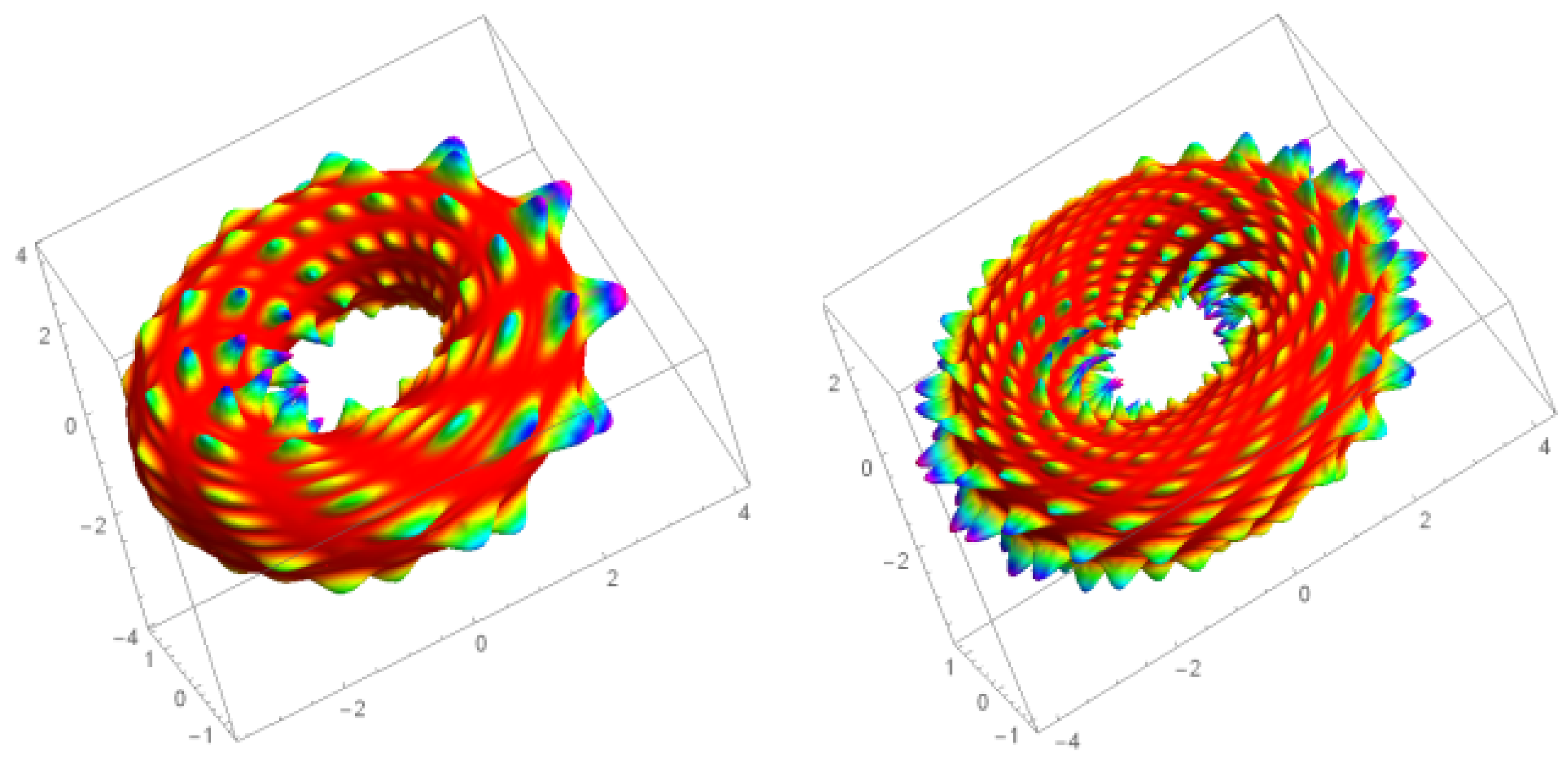


Interference Patterns for 3 pairs of primes: 17 & 29, 17 & 19, and 41 & 43 (left to right). Note the distinct pattern when the primes are twins.

- Application to $\mathbb{Z}[i]$: In the three-dimensional case, similar interference patterns between Gaussian "twin" primes are observed.
- We encountered three definitions for twin primes in the Gaussian integers: Two Gaussian primes p_1, p_2 are called Gaussian "twin" primes if (1) $p_1 - p_2 = 1 + i$; (2) $p_1 - p_2 = 2$; (3) $p_1 - p_2 = 2i$.



A Gaussian twin interference pattern.



Interference patterns between Gaussian twin primes, on tori.

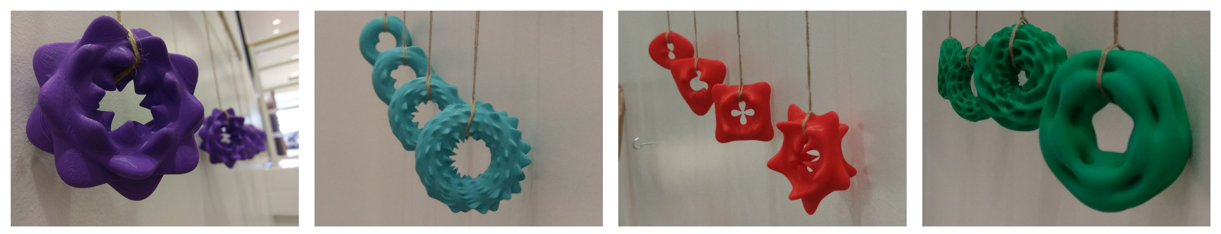
- Based on the interference patterns, we found the first definition to be more analogous to the definition of twin primes in the integers.

Weierstrass Functions

- After observing the results from mapping the periodic functions to the tori, we wanted to see what the Weierstrass Elliptic functions looked like mapped onto a torus since they are periodic as well.
- The problem though is that these are complex valued functions so we take the norm of the elliptic function to get a real valued function. Still though, this function has poles that go off to infinity so we take the arctangent of the norm in order to limit the function.

Description of Tori

The following key provides a description of the torus families featured in the hanging mobiles, according to color:



- **Purple:** Gaussian primes visualized using periodic functions mapped onto the surface of tori.
- **Blue:** Interference patterns of the periodic functions that are representative of various Gaussian twin primes.
- **Red:** Weierstrass elliptic functions mapped onto the surface of tori.
- **Green:** Inverted functions selected from the three previous families of tori.