Graph Theory - Solutions

November 18, 2015

1 Warmup: Cycle graphs

Definition 1. *The* cycle graph C_n *is the graph given by the following data:*

$$\begin{aligned} V_G &= \{v_1, v_2, \ldots, v_n\} \\ E_G &= \{e_1, e_2, \ldots, e_n\} \\ \epsilon(e_i) &= \{v_i, v_{i+1}\}, \end{aligned}$$

where the indices in the last line are interpreted modulo n.

1. Draw C_n for $n = 0, 1, 2, 3, 4, 5$.

   Soln. These look like loop graphs, or bracelets. The $n = 0$ graph is empty, the $n = 1$ is a single vertex with a loop on it, and $n = 2$ is two vertices with a double edge between. $n = 3$ is a triangle, $n = 4$ is a square, etc.

2. Is C_n simple?

   Soln. For $n \geq 3$, yes. I suppose technically for $n = 0$ too. But $n = 1$ has a loop and $n = 2$ has a double edge.

3. Is C_n bipartite?

   Soln. If n is even, yes. If n is odd, no.

4. Does C_n have an Eulerian circuit? How many?

   Soln. Yes. You can go around clockwise or counterclockwise from any single vertex, making a full loop around the cycle. So there are $2n$.

5. Does C_n have a Hamiltonian cycle? How many?

   Soln. Yes, $2n$ as above (the same walks solve this problem also).

6. Is C_n connected? Prove it (from the definition).

   Soln. Yes. Let v_i and v_j be vertices. Then there is a walk from v_i to v_j given by $v_i e_i v_{i+1} e_{i+1} \cdots e_{j-1} e_j$. (Note that the indices are interpreted modulo n.)

7. How many walks of length 3 are there in C_n?

   Soln. $8n$. For a walk, you can repeat. So from your starting vertex (of which there are n choices), you may walk two directions. In the next step you may again walk two directions. And for the last step, you may again walk two directions. None of the resulting $8n$ walks are the same.

8. How many paths of length 3 are there?

   Soln. $2n$. Choose a starting vertex (n choices), then walk one of two directions. Now since you can't repeat vertices, you have no more choices in the remaining two steps: you must continue in the direction you set out.

9. How many trails of length 3 are there?

   Soln. $2n$, as for the last question.

10. Compute the adjacency matrix A of C_n.

    Soln. The main diagonal of the matrix has zeroes. The diagonal directly above and below this have 1s. The top right and lower left corner have 1s. The rest is zeroes. For example,

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

    Note that since each vertex is degree 2, each column or row should have exactly two 1s in it.

11. Without doing any matrix multiplications, compute A^{n-1}.

    Soln. This was harder than anticipated. The v_i, v_j entry should represent the number of walks from v_i to v_j of length $n-1$. So, for example, if $n-1$ is even, then the only walks are the 2 that go all the way around the cycle the long way. But if $n-1$ is odd, then you may have walks that go back on themselves a lot without going all the way around. So, this is doable but fiddly.

12. Suppose the edge e_i is given the weight i. What spanning tree does one obtain from the greedy algorithm?

    Soln. The subgraph which is just missing e_n.

13. How many spanning trees does C_n have (from the definition).

    Soln. You can miss any one edge to create a spanning tree. So there are n.

14. Prove the last item using the Matrix Tree Theorem we saw in class. (Hint: figure out a recursion for the determinants.)

    Soln. The Matrix Tree Theorem only applies to simple graphs, so require $n \geq 3$. It requires constructing a matrix with degrees on the diagonals and -1 in the i, j entry if and only if v_i is connected to v_j by an edge. In our case, this looks similar to the negative of the adjacency matrix,

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

We need to delete the last row and column. So it has the form

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

with $n-1$ rows and columns. The first few determinants ($n = 2, 3, 4$) are (by direct computation), $2, 3, 4$. It is fairly easy to show (using expansion on the last row) that the determinant D_n of the $(n-1) \times (n-1)$ matrix satisfies:

$$D_n = 2D_{n-1} - D_{n-2}$$

which, with the given initial values, has solution $D_n = n$. So there are n spanning trees.

2 Bipartite graphs

Definition 2. *The* complete bipartite graph $K_{n,m}$ *is the graph given by the following data:*

$$\begin{aligned} V_G &= \{v_1, v_2, \ldots, v_n\} \cup \{w_1, w_2, \ldots, w_m\} \\ E_G &= \{e_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\} \\ \epsilon(e_{i,j}) &= \{v_i, w_j\} \textit{ for all } 1 \leq i \leq n, 1 \leq j \leq m. \end{aligned}$$

1. Draw $K_{1,1}$, $K_{2,3}$, $K_{4,2}$ and $K_{0,1}$.

   Soln. Look up pictures on Wikipedia. Nicer than anything I can create in LaTeX right now.

2. Does $K_{n,m}$ have an Eulerian circuit?

   Soln. An Eulerian circuit exists if and only if every edge has even degree. The degree of the v_i is m and the degree of the v_j is n. So it does if and only if n and m are even.

3. Does $K_{n,m}$ have a Hamiltonian cycle?

   Soln. This is a little trickier. But colour each v_i red and each v_j blue. Then any Hamiltonian cycle will visit vertices alternating between red and blue. So in the final cycle, there will be the same number of reds as blues. Hence, the answer is NO unless $n = m$. If $n = m$, then you can explicitly construct one, say

   $$v_1 e_{1,1} w_1 e_{1,2} v_2 e_{2,2} w_2 \cdots w_n.$$

   That is, visit $v_1 w_1 v_2 w_2 \cdots w_n$ in that order; there is always a new edge when you need one. Note: this will not use all edges, so it is not an Eulerian circuit.

4. Compute the adjacency matrix of $K_{n,m}$.

   Soln. List the vertices in the order given in the definition. It is a block matrix. The matrix is $(n+m) \times (n+m)$, with blocks of size $n \times n$, $n \times m$, $m \times n$ and $m \times m$. The first and last blocks just listed are all zero. The other blocks are all filled with 1s. Example, $n = 2$, $m = 3$:

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

5. Compute the number of paths of length 1.

   Soln. This is just the number of edges, times two (for going each direction). That is $2nm$.

6. Compute the number of paths of length 2.

   Soln. A path can't repeat vertices. By our definition, it could be a closed path out-and-back along the same edge, of which there are $2nm$ (two for each edge). Besides these, starting at any v_i there are $m(n-1)$ and starting at a w_i there are $m(n-1)$. So in total, then $2nm+nm(n-1)+mn(m-1)$, where you should ignore the first term if you are using the book's definition of paths (in which edges cannot be repeated).

7. Prove that $K_{n,m}$ is connected. (Any method.)

   Soln. We will show there is a walk between any two vertices. To get from v_i to w_j use $e_{i,j}$. To get from v_i to v_j use $e_{i,1}e_{1,j}$. Finally, to get from w_i to w_j use $e_{1,i}e_{1,j}$.

8. Count the number of spanning trees of $K_{n,m}$.

   Soln. This is done in your book, Example 10.22.