Game Theory: A Few First Games

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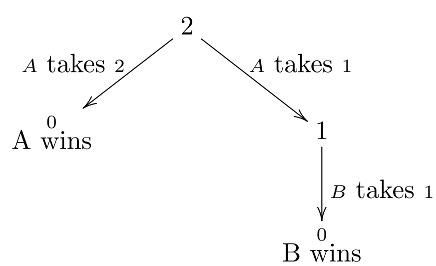
1 Taking One or Two

Play this game with a partner. To set up the board, place any number n of chips in a row (the arrangement doesn't matter, just the number of chips). Players take turns. On your turn, you may remove either 1 or 2 chips from the row (it doesn't matter which ones; all chips are identical). The last player to take chips (leaving an empty row) wins.

1. Play the game until you understand the rules.

2. This is a deterministic game (no chance). Each player has two choices on his turn. Therefore, we can draw a *game tree* showing all the possible ways the game may play out. For example, the game tree for a game of 2 chips looks like this:

   If the first player plays intelligently (for the game of 2 chips), can he always win?

3. Draw the game tree for 3 chips. Can the first player always win, if he plays it right?

4. Draw the game tree for 4 chips. Can the first player always win?

5. Here's the motivating definition for us for today.

   Definition 1. *The game of n chips is a* winning position *if the first player can always win, no matter what his opponent does, provided he makes the right moves.*

   Make a chart showing whether the game of n chips is a winning position or not, for $n = 1, 2, 3, \ldots$. How far can you keep filling out this chart? How can you use previous information to fill out this chart efficiently? (Definitely don't draw out a new game tree for each n !)

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16 ...
win?																

6. Make a conjecture of the form 'The game of n chips is a winning position if and only if ...'

7. Assuming your conjecture is right, explain the winning strategy for playing this game. Test out your winning strategy by playing the game a couple more times.

8. In filling out the chart in the last part, you were using information for previous columns of the chart to figure out the current column of the chart. Can you formalize this into a proof?

2 Some more games

Solve the following games, in the sense of determining the winning positions and winning strategies, and proving it.

1. Suppose there are two piles of chips. On each turn, a player may remove any positive number of items from one of the piles (the player may not remove items from more than one pile). *Hint: Possible positions consist of two numbers (the sizes of the two piles); consider these as Cartesian coordinates, to help with visual thinking.*

2. Suppose there are two piles of chips. On each turn, a player may remove either 1 or 2 chips from one of the piles (the player may not remove items from more than one pile).

3. Suppose there are two piles of chips. On each turn, a player may remove anywhere from 1 to n chips from one of the piles (the player may not remove items from more than one pile).

4. Suppose there are m piles of chips. On each turn, a player may remove any positive number of items from one of the piles (the player may not remove items from more than one pile).