Worksheet on Existence Proofs

October 30, 2015

1 Proof of existence

1. Prove that there exists an integer satisfying $x^2 = x$.

2. Prove that for every non-zero real number x, there exists a real number y such that $x^2y = 1$

2 Irrationals

Without reference to your notes from Wednesday (!), re-create the proof that between every two distinct rational numbers, there is an irrational number.

Here are some hints as to how to proceed:

1. Show that there is an irrational number between 1 and 100

2. Show that there is an irrational number between -100 and 0

3. Show that there is an irrational number between 100 and 200

4. Show that there is an irrational number between 0 and 1

5. Show that there is an irrational number between 0 and $1/100$

6. Let n be an integer. Show that there is an irrational number between 0 and $1/n$

7. Let n be an integer and a be a rational number. Show that there is an irrational number between a and $a + 1/n$.

8. Let a and b be distinct rational numbers, $a < b$. Show that there is an irrational number between a and b. (This is the full theorem.) Please write out this full theorem neatly.

3 Convergence of Sequences: Notes

Here's some review of what was done in class on Wednesday. Please fill in the blanks.

Definition 1. *A sequence $L_1, L_2, L_3, \ldots, L_n, \ldots$ of real numbers is said to* converge to $L \in \mathbb{R}$ *if, for every real $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|L_n - L| < \epsilon$ for all $n > N$. The number L is called* the limit *of the sequence.*

Please draw parentheses on the definition to indicate the structure, to help you read it.

Theorem 1. *The sequence $L_n = 1/n$ converges to 0.*

Let's restate the theorem (unpacking the definition of 'converges') to illustrate the 'for all, there exists' form.

Theorem 2. *Let $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that $|1/n - 0| < \epsilon$ for all $n > N$.*

The sequence starts $1, 1/2, 1/3, 1/4, 1/5, 1/6, \ldots$
Examples:

1. If $\epsilon = 2$, then all terms satisfy $|L_n - 0| < \epsilon$. So $N =$ ____________ suffices.
2. If $\epsilon = 3/4$, then all terms $n > 2$ satisfy $|L_n - 0| < \epsilon$. So $N =$ ____________ suffices.
3. If $\epsilon = 1/100$, then all terms $n > 100$ satisfy $|L_n - 0| < \epsilon$. So $N =$ ____________ suffices.

General pattern? Need $|L_n| = |L_n - 0| < \epsilon$, i.e. $1/n < \epsilon$. So take $n >$ ____________.

Proof. Let $\epsilon > 0$. Let N be any natural number greater than ____________. Then for any $n > N$, we have $n >$ ____________. Therefore $1/n < \epsilon$. Therefore $|1/n - 0| < \epsilon$. $\square$

4 Convergence of Sequences: Practice

1. For each sequence, determine the limit and give a proof that it converges to that limit.

   (a) $L_n = 2 - 1/n$

   (b) $L_n = 1/\sqrt{n}$

2. Prove that if L_n converges to A and M_n converges to B, then $L_n + M_n$ converges to $A + B$.