Counting!

October 16, 2015

In the packet you will find eight different counting problems. Each gives a sequence of integers. To get acquainted with each counting problem, compute the values for $n = 1, 2, 3$ at least, $n = 4$ if you dare.

Our questions for today are:

1. How do all these problems relate? Can you prove a relationship between different problems?

2. What formula(e) count(s) these objects? You can look for a formula, for a recurrence relation, or a generating function. Might I suggest looking for a recurrence relation in the parentheses, for example.

BIFURCATING TREES

How many rooted plane binary trees are there with n internal nodes? (Figure 4.5)

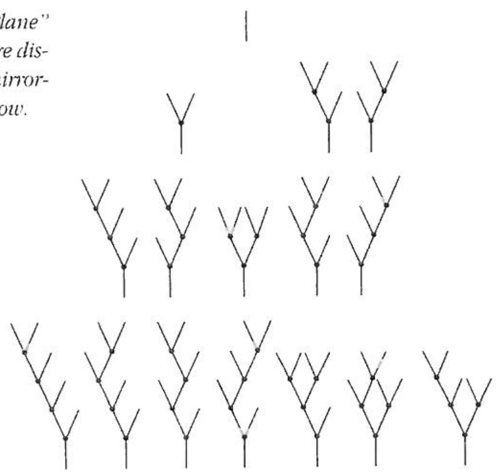

FIGURE 4.5 *Binary trees: "Plane" means that left and right are distinguishable. Add the mirror-images of trees in the last row.*

EVALUATING LADDERED EXPONENTS

How many values can you expect from an n-fold exponential? (Figure 4.6)

$$3^2 = 9$$

$$(4^3)^2 = 4^6, \quad 4^{(3^2)} = 4^9,$$

$$((5^4)^3)^2 = 5^{24}, \quad 5^{(4^{(3^2)})} = 5^{262144}, \quad (5^{(4^3)})^2 = 5^{128}, \quad (5^4)^{(3^2)} = 5^{36}, \quad 5^{((4^3)^2)} = 5^{4096}.$$

FIGURE 4.6 *Laddered exponents give varied values.*

ROOTED PLANE BUSHES

How many rooted plane bushes are there with n edges? (Figure 4.7)

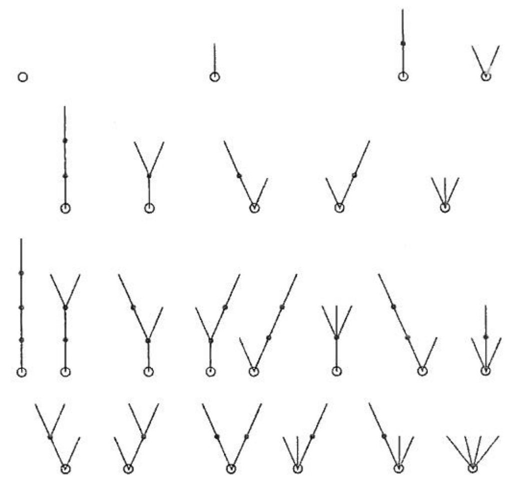

FIGURE 4.7 *Rooted plane bushes.*

MOUNTAINS

How many mountains can you draw with n upstrokes and n downstrokes? (Figure 4.8)

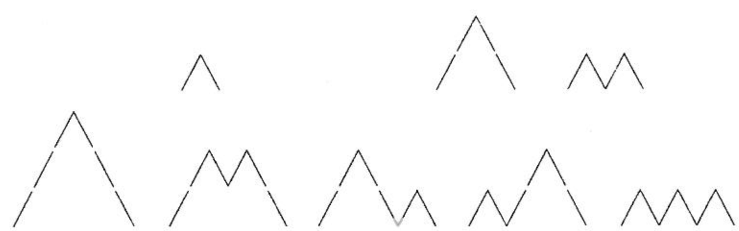

FIGURE 4.8 *Mountains.*

PARENTHESES

How many ways are there of arranging n pairs of parentheses? (Figure 4.9)

$n = 1$: ( ); $n = 2$: (( )), ( )( );

$n = 3$: ((( ))), (( )( )), (( ))( ), ( )(( )), ( )( )( )

FIGURE 4.9 *Arrangements of n pairs of parentheses.*

HANDS ACROSS THE TABLE

How many noncrossing handshakes are possible with n pairs of people? (Figure 4.10)

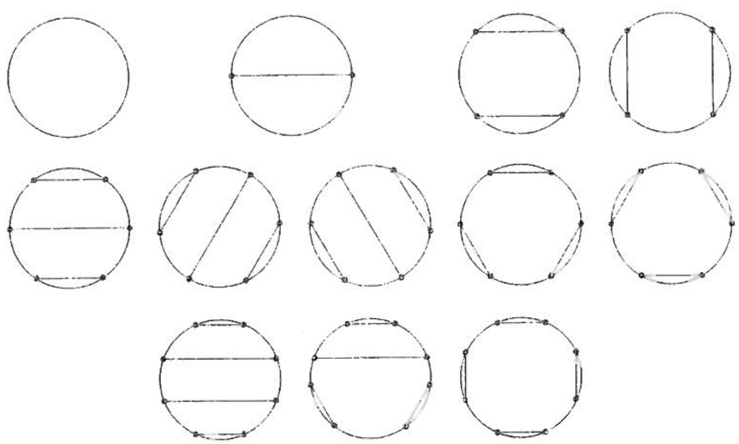

FIGURE 4.10 *Shaking hands without crossing. The last line gives 14 ways by rotating*

FRIEZE PATTERNS

How many different diagonals are possible in a frieze pattern with n + 1 rows? The answers for n = 1, 2, 3 are 1, 2, 5, respectively (Figure 4.3).

1 1 1 ...
1 1 1 ...

1 1 1 1 1 ...
1 2 1 2 1 ...
1 1 1 1 1 ...

1 1 1 1 1 1 1 ...
1 2 2 1 3 1 2 ...
1 3 1 2 2 1 3 ...
1 1 1 1 1 1 1 ...

FIGURE 4.3 *Frieze patterns with n + 1 rows.*

CHOPPING POLYGONS INTO TRIANGLES

How many ways are there of chopping a given (n+2)-sided polygon into n triangles? (Figure 4.4)

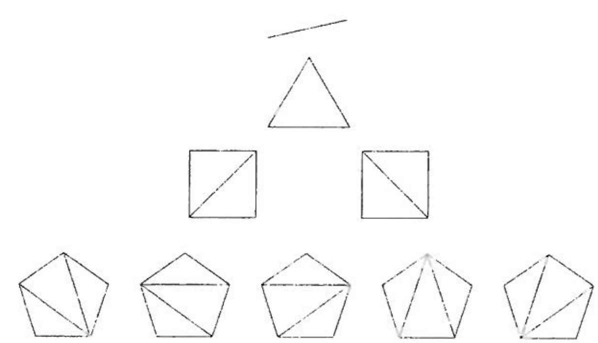

FIGURE 4.4 *Chopping polygons into triangles.*

Frieze Patterns

In Figure 3.8(a) we've drawn a pattern bounded by a zigzag of zeros at the left and horizontal lines of zeros above and below. In Figure 3.8(b) we've used ones instead of zeros. Now fill in the question marks by the rule that the numbers a and d in each little diamond

$$\begin{matrix} & b & \\ a & & d \\ & c & \end{matrix}$$

add to 1 more than do b and c in Figure 3.8(a), while they *multiply* to 1 more than do b and c in Figure 3.8(b).

Some surprising things happen, as shown in Figure 3.9(a) and (b). For the *additive* pattern, part (a), the next zeros in each line form a copy of the initial zigzag, so the pattern repeats itself every seven places. The *multiplicative* pattern, part (b), is even more surprising; all the divisions come out exactly, so that the entries are whole numbers. This time the ones in each row form an *upside-down* copy of the initial zigzag. We have to go a total of nine places in each row before we get an exact repetition.

The same sort of thing happens for arbitrary widths and shapes of initial zigzag, as you can verify by experiment. See if you can work out why.

For multiplicative frieze patterns, the essential observation is that for any six entries such as

$$\begin{matrix} & b & & \\ a & & d & \\ & c & & f \\ & & e & \end{matrix}$$

we have $(a + e)/c = (b + f)/d$. Figure 3.10 shows how this implies that a number x just above the lower row of ones will reappear sometime later, just below the upper row of ones.

There are other ways of starting than by using a zigzag of 1s. In fact, you can use any diagonal sequence of numbers

$$1 = a_0, a_1, \ldots, a_n = 1 \text{ such that } a_i \text{ divides } a_{i-1} + a_{i+1}.$$

How many such sequences are there? We'll tell you the answer in the next chapter.

(a)

```
0 0 0 0 0 0 0 0 0
 0 ? ? ? ? ? ? ?
  0 ? ? ? ? ? ? ?
   0 ? ? ? ? ? ? ?
  0 ? ? ? ? ? ? ?
 0 ? ? ? ? ? ? ?
  0 ? ? ? ? ? ? ?
   0 0 0 0 0 0 0 0
```

(b)

```
1 1 1 1 1 1 1 1 1
 1 ? ? ? ? ? ? ?
  1 ? ? ? ? ? ? ?
   1 ? ? ? ? ? ? ?
  1 ? ? ? ? ? ? ?
 1 ? ? ? ? ? ? ?
  1 ? ? ? ? ? ? ?
   1 1 1 1 1 1 1 1
```

FIGURE 3.8 *Fill in these friezes, using the diamond rule.*

(a)

```
0 0 0 0 0 0 0 0
 0 1 2 5 6 4 3 0
  0 2 6 10 9 6 2 0
   0 5 10 12 10 4 1 0
  0 2 8 11 12 7 2 0
 0 1 4 8 10 8 4 0
  0 2 3 6 5 4 1 0
   0 0 0 0 0 0 0 0
```

(b)

```
1 1 1 1 1 1 1 1 1 1
 1 2 2 4 2 1 3 2 4 1
  1 3 7 7 1 2 5 7 3 1
   1 10 12 3 1 3 17 5 2 1
  1 3 17 5 2 1 10 12 3 1
 1 2 5 7 3 1 3 7 7 1
  1 3 2 4 1 2 2 4 2 1
   1 1 1 1 1 1 1 1 1 1
```

FIGURE 3.9 *Filled in frieze patterns repeat after so many steps.*

THE BOOK OF

Numbers

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