An invitation to category theory

Note: The answers to this are especially easy to find online or elsewhere, so please take a stab at it before you google anything.

Let $\mathcal{C} = (\mathcal{O}, \mathcal{M})$ be a category, and let X_1 and X_2 be objects.

Definition 1. *An object X is called the* product *of X_1 and X_2, denoted $X_1 \times X_2$ if it has two morphisms $\pi_1 : X \to X_1$ and $\pi_2 : X \to X_2$ with the following property.*

- *(Property A) For any object Y and morphisms $f_1 : Y \to X_1$ and $f_2 : Y \to X_2$, there exists a unique morphism $f : Y \to X$ such that $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$.*

1. Show that the product of two objects, if it exists, is unique up to isomorphism.
2. Determine whether the product exists and what it is, in the case of the category of sets. By 'what it is' I mean give an explicit construction of the object, or identify it by a name you already know. Please prove the object does in fact have the property.
3. Repeat the last part for the category of groups.

By 'reversing arrows,' we get the definition of a coproduct.

Definition 2. *An object X is called the* coproduct *of X_1 and X_2, denoted $X_1 \oplus X_2$ if it has two morphisms $i_1 : X_1 \to X$ and $i_2 : X_2 \to X$ with the following property.*

- *(Property B) For any object Y and morphisms $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$, there exists a unique morphism $f : X \to Y$ such that $f_1 = f \circ i_1$ and $f_2 = f \circ i_2$.*

1. Show that the coproduct of two objects, if it exists, is unique up to isomorphism.
2. Determine whether the coproduct exists and what it is, in the case of the category of sets. Prove it, as above.
3. Repeat the last part for the category of groups.

Optional, no credit: Consider some other categories you know, like vector spaces or rings. Or even topological spaces (morphisms are continuous maps).

Another optional, no credit: There is a way in which you can view groups as categories having a single object and invertible morphisms. Explain.