Chapters 5,6,7 Review
SOLUTIONS PROBLEMS 36-47
Math 52 Spring 2006

36. (a) Find bases for the subspaces $\ker A$ and $\operatorname{im} A$ associated to the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 15 \\ 3 & 14 & 25 & -3 \end{bmatrix}.$$

(b) What is the orthogonal complement of the kernel of the above matrix A? Verify orthogonality.

(a) The reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & -1 & \frac{5}{2} \\ 0 & 1 & 2 & \frac{-3}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

from whence we deduce that

$$\ker A = span \left\{ \begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ -3 \\ 0 \\ 4 \end{bmatrix} \right\}$$

$$\operatorname{im} A = span \left\{ \begin{bmatrix} 1 \\ 5 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 10 \\ 14 \end{bmatrix} \right\}$$

(b) To find the orthogonal complement, we must find all vectors perpendicular to the vectors $\begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ -3 \\ 0 \\ 4 \end{bmatrix}$. That is, we must solve the system $B\vec{x} = \vec{0}$ where

$$B = \begin{bmatrix} -1 & 2 & -1 & 0 \\ -10 & -3 & 0 & 4 \end{bmatrix}$$

This matrix has the following kernel

$$(\ker(A))^{\perp} = \ker(B) = \mathit{span}\left\{ \begin{bmatrix} -3 \\ 10 \\ 23 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 4 \\ 0 \\ 23 \end{bmatrix} \right\}$$

37. Find an orthonormal basis for the subspace W of $\mathbb{R}^4$ spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

Gram-Schmidt gives

$$\vec{u}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \vec{u}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

38. (a) Find the orthogonal projection of the vector $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ onto the image of the matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 2 & 2 \end{bmatrix}$.

(b) Find a basis for $(\operatorname{im} A)^{\perp}$, i. e. the orthogonal complement of $\operatorname{im} A$ in $\mathbb{R}^3$.

(a) Let

$$\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}.$$

Then, normalizing, we have

$$\vec{u}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \vec{u}_2 = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}.$$

So

$$\begin{aligned} \text{proj}_{im(A)}(\vec{v}) &= (\vec{v} \cdot \vec{u}_1)\vec{u}_1 + (\vec{v} \cdot \vec{u}_2)\vec{u}_2 \\ &= (0)\vec{u}_1 + (1)\vec{u}_2 \\ &= \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \end{aligned}$$

(b) To find the orthogonal complement, we must find all vectors perpendicular to the vectors $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$. That is, we must solve the system $B\vec{x} = \vec{0}$ where

$$B = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$$

This matrix has the following kernel

$$(\text{im}(A))^{\perp} = \ker(B) = span\left\{\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}\right\}$$

39. (a) Compute the determinant of the matrix $\begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}$.

(b) Find all values of t for which the matrix A is invertible.

$$A = \begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix}.$$

(a) It can be seen by inspection that this matrix has less than full rank. So its determinant should be zero. We can compute using row operations.

$$\begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix} \to \begin{bmatrix} 11 & 12 & 13 & 14 \\ 10 & 10 & 10 & 10 \\ 20 & 20 & 20 & 20 \\ 30 & 30 & 30 & 30 \end{bmatrix} \quad \text{2nd, 3rd, 4th minus 1st}$$

$$\to \begin{bmatrix} 11 & 12 & 13 & 14 \\ 10 & 10 & 10 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{r} \text{4th minus 2 x 2nd} \\ \text{3rd minus 2nd} \end{array}$$

Since it row reduces to obtain zero rows, the determinant is 0.

(b) The determinant of A (expanding down first column) is $1(1-t^2) - t(t-t^3) + t^2(0) = t^4 - 2t^2 + 1 = (t^2-1)^2$. Hence A is invertible as long as $t \neq 1, -1$.

40. (a) Compute the characteristic polynomial, the eigenvalues and corresponding eigenspaces of the matrix

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & -6 \\ 1 & 0 & -1 \end{bmatrix}.$$

(b) Is A diagonalizable? If not, explain why. If it is, write it as $A = PDP^{-1}$.

(a) Expand the determinant down the middle column for the characteristic polynomial. One obtains $(2-\lambda)[(1-\lambda)(-1-\lambda) - 3] = (2-\lambda)(\lambda-2)(\lambda+2)$. The eigenvalues are $2, 2, -2$.

$$E_2 = \ker \begin{bmatrix} -1 & 0 & 3 \\ 2 & 0 & -6 \\ 1 & 0 & -3 \end{bmatrix} = span \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$E_{-2} = \ker \begin{bmatrix} 3 & 0 & 3 \\ 2 & 4 & -6 \\ 1 & 0 & 1 \end{bmatrix} = span \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}$$

(b) It is diagonalizable, since the geometric multiplicities equal the algebraic multiplicities, so there are sufficient eigenvectors to form a basis. Putting this basis into a matrix, we obtain

$$P = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

So $A = PDP^{-1}$ where

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

41. (a) Let A be a 3×3 matrix having the following properties:

i. $\ker A$ contains the vector $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$;

ii. $A^3\vec{v} = 8\vec{v}$, where $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$;

iii. multiplication by A leaves every vector on the line lying in the horizontal (x_1, x_2)-plane of equation $x_1 + x_2 = 0$ unchanged.

Find the eigenvalues and eigenvectors of A. Is A diagonalizable? Is A invertible?

(b) Find a basis for $\operatorname{im} A$.

(a) This is a very advanced problem. Part (i) tell us that A has eigenvector $\vec{u}$ with eigenvalue 0. Since the anything in the kernel of A is also in the kernel of A^3, this also tells us A^3 has eigenvector $\vec{u}$ with eigenvalue 0 . Part (iii) tells us that A has eigenvector $\vec{w} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ with eigenvalue 1. Since if A leaves $\vec{w}$ unchanged, then A^3 must also, this also tells us that A^3 has eigenvector $\vec{w}$ with eigenvalue 1. Part (ii) tells us that A^3 has eigenvector $\vec{v}$ with eigenvalue 8. This tells us that A^3 has an eigenbasis made up of $\vec{u}, \vec{v}, \vec{w}$.

Now, A has eigenvalues 1 and 0 so its characteristic polynomial (of degree three) must factor completely and it has a third root. That third root cannot be 0, since then the kernel of A would be dimension two, and so the kernel of A^3 would also be dimension at least two (which it is not by the algebraic multiplicity of 0 as an eigenvalue for A^3). The third root also cannot be 1 since then A would leave fixed a two-dimensional subspace and so would A^3 (which would mean 1 would be an eigenvalue of algebraic multiplicity at least 2 for A^3, which it is not). So the third root is distinct from 0 and 1 and hence A has three distinct eigenvalues and is therefore diagonalizable with eigenbasis $\vec{u}, \vec{v}, \vec{w}$ and eigenvalues $0, 2$ and 1 respectively. (Note that once we know $\vec{v}$ is an eigenvector for A, then it must have eigenvalue 2 for A since it has eigenvalue 8 for A^3.)

A is not invertible since it has nontrivial kernel.

(b) The matrix A has nullity 1 so the image must have dimension 2. We have $\operatorname{im}(A) = span\{A\vec{u}, A\vec{v}, A\vec{w}\} = span\{\vec{v}, \vec{w}\}$.

42. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 4 & 6 \\ 1 & 4 & 6 \end{bmatrix}$.

(a) Find an orthonormal basis q_1, q_2, q_3 for the image of A.

(b) Find a vector q_4 such that q_1, q_2, q_3, q_4 is an orthonormal basis for $\mathbb{R}^4$.

(a) Applying the Gram-Schmidt process to the columns of A yields

$$\vec{q}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{q}_2 = \frac{1}{2\sqrt{3}}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{q}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}.$$

(b) To do this, one could find a vector independent from the first three (perhaps by inspection), and then apply Gram-Schmidt. However, it is easy to guess a solution by inspection. To by systematic about it, note that a vector orthogonal to $\vec{q}_1, \vec{q}_2, \vec{q}_3$ must have dot product zero with all three, i.e., it is in the kernel of the matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 1 & 1 & 1 \\ 0 & -2 & 1 & 1 \end{bmatrix}$$

A possible vector is $q_4 = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ (note that it has been normalized to length one).

43. Compute the following determinant:

$$\det \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ -1 & -6 & 1 & -1 \end{bmatrix}.$$

We can do this by row-reduction operations easiest. The solution is $\det = -4$.

44. Let V be the plane in $\mathbb{R}^3$ defined by $x - 2y + z = 0$.

(a) Find a basis for V.

(b) Find a basis for the orthogonal complement $V^{\perp}$ of V.

(c) Find the matrix P of the projection onto V.

(d) Find all the eigenvalues and eigenvectors of P. *Hint: P is a projection matrix.*

(a) This is simple by inspection. A possible basis (among many correct answers!) would be

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

(b) The orthogonal complement of a plane in $\mathbb{R}^3$ is its normal line. So a vector perpendicular to the plane is (from the plane equation coefficients):

$$\vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

(c) Since $\vec{v}_1$ and $\vec{v}_2$ are not orthogonal, we can use Gram-Schmidt to make them orthogonal. Alternatively, we can replace v_2 with the cross-product of v_1 and v_3 to get a vector orthogonal to these two. I will choose this latter approach, and then normalising, we have

$$\vec{u}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}.$$

Thus, we define the matrix

$$Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{14}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{14}} \\ 0 & \frac{-3}{\sqrt{14}} \end{bmatrix}$$

and then the matrix of the desired projection is

$$QQ^T = \begin{bmatrix} \frac{61}{70} & \frac{18}{70} & \frac{-3}{14} \\ \frac{18}{70} & \frac{34}{70} & \frac{6}{14} \\ \frac{-3}{14} & \frac{6}{14} & \frac{9}{14} \end{bmatrix}$$

You could also do this using methods of Chapter 2.

(d) The eigenspace with eigenvalue 1 is $span\{\vec{v}_1, \vec{v}_2\} = span\{\vec{u}_1, \vec{u}_2\}$. The eigenvector with eigenvalue 0 is $\vec{v}_3$.

45. Consider the matrix $A = \begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \\ 1 & 0 \end{bmatrix}$.

(a) Diagonalize A, i. e. find an invertible matrix S and a diagonal matrix D such that $D = SAS^{-1}$.

(b) Calculate $A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(a) The characteristic polynomial of A is $(\lambda - 1)(\lambda - \frac{1}{3})$. So the eigenvalues are 1 and $\frac{1}{3}$. The associated eigenspaces are

$$E_1 = \ker \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ 1 & -1 \end{bmatrix} = span \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$E_{\frac{1}{3}} = \ker \begin{bmatrix} 1 & -\frac{1}{3} \\ 1 & -\frac{1}{3} \end{bmatrix} = span \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

It is diagonalisable and

$$S = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$$

So $A = SDS^{-1}$ where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

(b) We have

$$
\begin{aligned}
A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= SD^nS^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3}^n \end{bmatrix} \left(\frac{-1}{2}\right) \begin{bmatrix} 1 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \left(\frac{-1}{2}\right) \begin{bmatrix} 1 - 3^{1-n} \\ 1 - 3^{-n} \end{bmatrix}
\end{aligned}
$$

46. Consider

$$
A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 7 & 9 & 11 & 13 \\ 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}
$$

(a) Is A invertible? Why or why not? Is A^TA invertible? Why or why not?

(b) Write down a formula for the projection onto the subspace spanned by the columns of A. (Do not multiply out!)

(a) A is not invertible since it is not square. Note that A corresponds to a transformation from $\mathbb{R}^6$ to $\mathbb{R}^4$. And A^T is a transformation from $\mathbb{R}^4$ to $\mathbb{R}^6$. Therefore A^TA corresponds to the composition of these two transformations. Since the first one must have nontrivial kernel, the composition must have nontrivial kernel and hence is not invertible.

(b) There are some redundant columns in A. We remove them, to obtain

$$
\operatorname{im}(A) = \mathit{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 11 \\ 2 \\ 1 \end{bmatrix} \right\}
$$

To obtain the formula, you would first Gram-Schmidt these vectors and then put them as columns into a matrix Q and the projection will be QQ^T. This question was designed with a faster method in mind that we didn't study in Spring 2006.

47. Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 9 \\ 0 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

The determinant is 3.