Chapters 5,6,7 Review
SOLUTIONS PROBLEMS 21-35
Math 52 Spring 2006

21. Compute the determinant of the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 6 & 4 \\ 0 & 1 & 13 & 1 \\ 0 & 0 & -2 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Is A invertible or not? Why?

The determinant of the three matrices can be done individually. For the first, it can be reduced to the identity matrix by one row swap, so it has determinant -1. The second is upper triangular and its determinant is the product of the diagonal entries: $(3)(1)(-2)(4) = -24$. The third is lower triangular and has determinant $(1)(1)(1)(1) = 1$. Therefore the determinant of A is 24.

Since the determinant is nonzero, A is invertible.

22. Let $A = \begin{bmatrix} 2\sqrt{2} & \sqrt{6} & \sqrt{2} \\ -2\sqrt{2} & \sqrt{6} & \sqrt{2} \\ 0 & -2 & 2\sqrt{3} \end{bmatrix}$.

(a) Compute AA^T.

(b) What is A^{-1}?

(a) $AA^T = 16I$

(b) $A^{-1} = \frac{1}{16}A^T$

23. True or false? The product of any two orthogonal matrices is orthogonal.

    True. An orthogonal matrix is the matrix of an orthogonal transformation. Orthogonal transformations preserve length. The product of two orthogonal matrices is the composition of two orthogonal transformations, and since each preserves length, so does their composition. Therefore the product of two orthogonal matrices is orthogonal.

24. Define each of the following terms and in each case give a 2×2 example:

    (a) diagonal matrix

    (b) upper triangular matrix

    (c) diagonalizable matrix

    (a) A diagonal matrix is a square matrix whose off-diagonal entries are all zero. For example, $\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$.

    (b) An upper triangular matrix is a square matrix whose entries below the diagonal are all zero. For example, $\begin{bmatrix} 3 & 7 \\ 0 & -2 \end{bmatrix}$.

    (c) A diagonalizable matrix is a square matrix A such that $A = SDS^{-1}$ for some invertible matrix S and diagonal matrix D. For example,

    $$A = \begin{bmatrix} -2 & 3 \\ -4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

    Note that one can create such examples by choosing any diagonal D and invertible S and calculating A.

25. Let A be an $n \times n$ matrix.

    (a) Suppose $\vec{v}$ is an eigenvector of A with eigenvalue λ. Show that $\vec{v}$ is also an eigenvector of A^2.

    (b) Prove that if A is diagonalizable, then so is A^2.

(a) Suppose $\vec{v}$ is an eigenvector of A with eigenvalue λ. Then, $A\vec{v} = \lambda\vec{v}$. So $A^2\vec{v} = AA\vec{v} = A\lambda\vec{v} = \lambda A\vec{v} = \lambda^2\vec{v}$. This shows that $\vec{v}$ is an eigenvector of A^2 with eigenvalue λ^2.

(b) If A is diagonalizable, then $\mathbb{R}^n$ has a basis of eigenvectors of A. These eigenvectors are all eigenvectors of A^2 and so form a basis of eigenvectors that will also diagonalize A^2.

26. Let A and B be two similar matrices.

(a) Show that A and B have the same characteristic polynomial.

(b) Prove that A and B have the same eigenvalues.

(a) Suppose that A and B are two similar matrices. Then $A = SBS^{-1}$ for some invertible S. Then

$$\begin{aligned} A - \lambda I &= SBS^{-1} - \lambda I \\ &= SBS^{-1} - \lambda SIS^{-1} \\ &= S(B - \lambda I)S^{-1} && \text{by distributivity} \end{aligned}$$

Therefore $A - \lambda I$ and $B - \lambda I$ are similar. Since similar matrices have the same determinant, A and B have the same characteristic polynomial.

(b) Since A and B have the same characteristic polynomial, they have the same eigenvalues.

27. Decide if the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalizable. Justify your answer.

The matrix has exactly one eigenvalue with algebraic multiplicity three (since it is upper triangular, its diagonal entries give the eigenvalues). Therefore it will be diagonalizable if and only if the eigenvalue 1 has geometric multiplicity 3. But the matrix $A - I = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ has rank 1 and so has nullity 2. So it is not diagonalizable.

28. Find an orthogonal basis for the subspace of $\mathbb{R}^4$ spanned by

$$\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} \text{ and } \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}.$$

Using Gram-Schmidt, one obtains

$$\vec{u}_1 = \frac{1}{\sqrt{12}} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_2 = \frac{1}{\sqrt{12}} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_3 = \frac{1}{\sqrt{12}} \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

29. Find the eigenvalues of $\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. *Hint: they are integers.*

The eigenvalues are 2, 2 and 9.

30. The matrix $A = \begin{bmatrix} 4 & 3 & 3 \\ -12 & -8 & -6 \\ 6 & 3 & 1 \end{bmatrix}$ is diagonalizable and has eigenvalues $-2, -2, 1$. Find a matrix which diagonalizes A.

The eigenspaces are

$$E_{-2} = \ker \begin{bmatrix} 6 & 3 & 3 \\ -12 & -6 & -6 \\ 6 & 3 & 3 \end{bmatrix} = span \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right\}$$

$$E_1 = \ker \begin{bmatrix} 3 & 3 & 3 \\ -12 & -9 & -6 \\ 6 & 3 & 0 \end{bmatrix} = span \left\{ \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right\}$$

So $A = SDS^{-1}$ where $S = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 0 & 2 \\ 0 & -2 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

31. Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Using Gram-Schmidt, we obtain

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \frac{1}{2\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix}.$$

32. Let V be the vector space consisting of all polynomials $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ of degree ≤ 3. Let T be the linear transformation

$$T\left(p(x)\right) = x^2\,\frac{d^2p}{dx^2} + p(x).$$

(a) Find the matrix A associated to T for some suitable basis of V.

(b) For which real numbers λ does there exist a non-zero solution $p(x)$ to the equation

$$x^2\,\frac{d^2p}{dx^2} + p(x) = \lambda p(x)?$$

For each such λ find the corresponding $p(x)$.

(a) Use the basis $\mathcal{B} : 1, x, x^2, x^3$. Then

$$T(a+bx+cx^2+dx^3) = 2cx^2+6dx^3+a+bx+cx^2+dx^3 = a+bx+3cx^2+7dx^3$$

So the transformation has matrix

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

(b) The question is asking for the eigenvalues of the transformation T. Since the matrix is already diagonal, we see the eigenvalues are 1, 3 and 7. To find the eigenpolynomials associated to the eigenvalues, we could calculate the eigenspaces directly. However, since this matrix is already diagonal, the basis $\mathcal{B}$ is our eigenbasis and so we have

$$E_1 = span\{1, t\}$$
$$E_3 = span\left\{t^2\right\}$$
$$E_7 = span\left\{t^3\right\}$$

33. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$, find all numbers c for which the equation $A\vec{x} = cB\vec{x}$ has a nonzero solution. For each such c, find the corresponding $\vec{x}$.

We rearrange the equation by multiplying on the left by B^{-1}. We get

$$B^{-1}A\vec{x} = c\vec{x}$$

so the values of c we seek are the eigenvalues of the transformation $B^{-1}A = \frac{-1}{4}\begin{bmatrix} 0 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{4} & 0 \end{bmatrix}$

This transformation has characteristic polynomial $(\frac{1}{2} - c)(-c) - \frac{1}{4} = c^2 - \frac{1}{2}c - \frac{1}{4}$. This one requires the quadratic formula. The answers are $c = \frac{1 \pm \sqrt{-3}}{4}$. Since these are complex numbers, this problem really should not have been included in the review sheet. Nevertheless, the setup is interesting, and from here on, had there been real eigenvalues, the process would have been routine.

34. Let $\vec{v}$ and $\vec{w}$ be eigenvectors of A with corresponding eigenvalues 2 and 3, respectively. Are $\vec{v}$ and $\vec{w}$ linearly dependent or linearly independent? Give a detailed explanation (or proof).

The vectors are linearly independent.

Proof: The vectors satisfy $T(\vec{v}) = 2\vec{v}$ and $T(\vec{w}) = 3\vec{w}$. Suppose that they were dependent. Then for some a and b, we have $a\vec{v} + b\vec{w} = \vec{0}$.

That is, $\begin{bmatrix} a \\ b \end{bmatrix}$ is in the kernel of the matrix $B = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}$. Then, applying T, we obtain $\vec{0} = T(\vec{0}) = T(a\vec{v} + b\vec{w}) = 2a\vec{v} + 3b\vec{w}$. So $\begin{bmatrix} 2a \\ 3b \end{bmatrix}$ is also in the kernel of B. But subtracting these (since the kernel is a subspace), $\begin{bmatrix} 0 \\ b \end{bmatrix}$ is in the kernel. So $b\vec{w} = \vec{0}$ and hence $\vec{w} = \vec{0}$. But this contradicts the assumption that $\vec{w}$ is an eigenvector, since eigenvectors are nonzero vectors. Hence $\vec{v}$ and $\vec{w}$ must be independent.

35. Check whether the following subset is a vector subspace: The set of all 2×2 matrices with determinant equal to 0.

This is not a subspace since it fails to be closed under addition. As an example, the matrices $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are in the subspace but their sum is the identity, which does not have determinant 0. Note that the subspace is closed under scalar multiplication and it includes the zero matrix.