Chapters 5,6,7 Review
Math 52 Spring 2006

This packet contains review problems for chapters 5, 6 and 7, and some problems relating the various parts of the course (hence covering earlier material). Please see previous packets for review problems relating to earlier chapters only. We also suggest below problems from the textbook for chapters 5, 6, and 7. (Problems from earlier chapters were suggested on earlier review packets.) The suggested problems are often higher-numbered, more conceptual problems. If you are having trouble with any of the more basic problems, you should pick some of those yourself to do.

Good luck studying!

Section 5.1: #17, 19, 29
Section 5.2: #32-35
Section 5.3: #33, 38, 52, 53, 56, 57
Chapter 5 True/False: #21, 23, 24
Section 6.1: #49, 50, 57
Section 6.2: #40, 41, 45, 48, 53
Chapter 6 True/False: #10-13, 18, 35
Section 7.2: #19, 38, 47, 48
Section 7.3: #23, 29, 31, 39
Section 7.4: #47, 51, 60, 64
Chapter 7 True/False: #4, 5, 6, 9, 11, 25, 29, 30, 32, 46, 53

1. (a) Express the matrix $A = \begin{bmatrix} 0.5 & 0 \\ 2 & 1.5 \end{bmatrix}$ as a product SDS^{-1}, where D is a diagonal matrix.

(b) Find a formula for $A^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

2. Compute the determinant of the following matrix:

$$\begin{bmatrix} 1 & -1 & -2 & 6 \\ 3 & 1 & 2 & 4 \\ 2 & 0 & 5 & 1 \\ -2 & 3 & 2 & 3 \end{bmatrix}.$$

3. Prove or disprove and salvage if possible:

   (a) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and define the transpose of A by $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Then A and A^T have the same eigenvalues.

   (b) Every 3×3 matrix has at least one real eigenvalue.

   (c) A real number λ is an eigenvalue of A if and only if λ is an eigenvalue of A^n for all positive integers n.

4. **Either** give an example exhibiting the stated properties **or** prove that no such example exists.

   (a) Square matrices A and B with the same characteristic polynomial so that A is not similar to B.

   (b) A square matrix A which is not diagonalizable.

5. Assume that

$$A = \begin{bmatrix} 3 & 4 & 3 \\ -1 & -4 & -5 \\ 1 & 8 & 9 \end{bmatrix}$$

   has characteristic polynomial $16 - 20t + 8t^2 - t^3 = -(t-2)^2(t-4)$. Find the eigenvalues and eigenspaces of A.

6. Let $T : P_2 \to P_2$ be defined by $T(f) = f + f' + f''$. Find an eigenbasis for T.

7. Let
$$\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$
These vectors form a basis of $\mathbb{R}^3$. (Note: you do not have to show this.)

   (a) Use the Gram-Schmidt process on these vectors to produce an orthonormal basis of $\mathbb{R}^3$.

   (b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection of $\mathbb{R}^3$ onto the subspace spanned by $\vec{v}_1$ and $\vec{v}_2$. Write down a matrix representing T. *Hint: your work in part (a) might be useful.*

8. Let
$$A = \begin{bmatrix} -2 & 5 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

   (a) Find the characteristic polynomial of A. What are the eigenvalues of A? *Hint: It factors!*

   (b) Find an invertible matrix S and a diagonal matrix D so that $A = SDS^{-1}$. *Hint: If this is painful or impossible, you may have found the wrong eigenvalues!*

9. Let
$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix}.$$
True or false?

   (a) A is invertible.

   (b) A has rank 2.

   (c) There exists a basis of eigenvectors for A.

10. Let

$$B = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

True or false? (Do these without calculating the characteristic polynomial.)

(a) B has rank 4.

(b) $\lambda = 0$ is an eigenvalue for B.

(c) $\lambda = 1$ is an eigenvalue for B.

(d) All eigenvalues of B satisfy $|\lambda| = 1$.

11. Let M_2 denote the vector space of all 2×2 matrices and $B = \begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$.

(a) Let T be the linear transformation from M_2 to M_2 defined by $T(C) = B^{-1}CB$. Consider the basis for M_2 consisting of

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The 4×4 matrix $A = \begin{bmatrix} 1 & & & 0 \\ 3 & & & -3 \\ 0 & & & 0 \\ 0 & & & 1 \end{bmatrix}$ represents T with respect to the basis $E_1, \ldots, E_4$. Supply the entries in the 2nd and 3rd columns of A.

(b) Find all numbers λ for which there exists a nonzero 2×2 matrix C with $B^{-1}CB = \lambda C$. *Hint: use the results in part (a). This is a chapter 7 problem!*

12. Let $V \subseteq \mathbb{R}^4$ be the subspace spanned by $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$.

Find an orthonormal basis for V.

13. Define what it means for a matrix to be **orthogonal**.

14. Let A be the matrix $A = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}$. Compute the eigenvalues and eigenvectors of A.

15. (a) A is a certain 3×3 matrix, which has three distinct real eigenvalues. Furthermore, two of its eigenvectors are $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Using only this information, find a third eigenvector for A which is not a linear combination of the above two.

    (b) Let $B = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}$. Find the eigenvalues of B.

16. (a) Compute the determinant of the matrix

$$C = \begin{bmatrix} 2 & 0 & -6 \\ 0 & -3 & 2 \\ 0 & 0 & -4 \end{bmatrix}.$$

    (b) Recall that a matrix Q is called **skew-symmetric** if $Q^T = -Q$. Prove that if Q is a 3×3 skew-symmetric matrix, then $\det Q = 0$.

    (c) Prove that if Q is any skew-symmetric matrix, than the trace of Q is 0. *Hint: what are the diagonal entries?*

17. (a) Let V be a vector space. Suppose $T : V \to V$ is a linear transformation with $T \circ T =$ Identity. Prove that all the eigenvalues of T are either 1 or -1.

    (b) Let V be the vector space of all 2×2 matrices. Let $T : V \to V$ be the linear map defined by $T(A) = A^T$. Find the eigenvalues and eigenmatrices of T. *Hint: use part (a)*

    (c) Let V and T be as in part (b). Write down a basis for V and find the matrix to describe T with respect to that basis.

18. Consider the matrix $A = \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & b \\ 1 & 1 & c \end{bmatrix}$.

    (a) Calculate the determinant of A.

    (b) Find a, b and c such that the image of A is $\mathbb{R}^3$.

19. Find all the eigenvalues of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Use one of the eigenvalues you found to calculate the associated eigenvectors.

20. True or false? Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then $A^{31} = A$.

21. Compute the determinant of the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 6 & 4 \\ 0 & 1 & 13 & 1 \\ 0 & 0 & -2 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

    Is A invertible or not? Why?

22. Let $A = \begin{bmatrix} 2\sqrt{2} & \sqrt{6} & \sqrt{2} \\ -2\sqrt{2} & \sqrt{6} & \sqrt{2} \\ 0 & -2 & 2\sqrt{3} \end{bmatrix}$.

    (a) Compute AA^T.

    (b) What is A^{-1}?

23. True or false? The product of any two orthogonal matrices is orthogonal.

24. Define each of the following terms and in each case give a 2×2 example:

    (a) diagonal matrix

(b) upper triangular matrix

(c) diagonalizable matrix

25. Let A be an $n \times n$ matrix.

    (a) Suppose $\vec{v}$ is an eigenvector of A with eigenvalue λ. Show that $\vec{v}$ is also an eigenvector of A^2.

    (b) Prove that if A is diagonalizable, then so is A^2.

26. Let A and B be two similar matrices.

    (a) Show that A and B have the same characteristic polynomial.

    (b) Prove that A and B have the same eigenvalues.

27. Decide if the matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalizable. Justify your answer.

28. Find an orthogonal basis for the subspace of $\mathbb{R}^4$ spanned by

$$\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} \text{ and } \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}.$$

29. Find the eigenvalues of $\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. *Hint: they are integers.*

30. The matrix $A = \begin{bmatrix} 4 & 3 & 3 \\ -12 & -8 & -6 \\ 6 & 3 & 1 \end{bmatrix}$ is diagonalizable and has eigenvalues $-2, -2, 1$. Find a matrix which diagonalizes A.

31. Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

32. Let V be the vector space consisting of all polynomials $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ of degree ≤ 3. Let T be the linear transformation

$$T\left(p(x)\right) = x^2 \frac{d^2 p}{dx^2} + p(x).$$

(a) Find the matrix A associated to T for some suitable basis of V.

(b) For which real numbers λ does there exist a non-zero solution $p(x)$ to the equation

$$x^2 \frac{d^2 p}{dx^2} + p(x) = \lambda p(x)?$$

For each such λ find the corresponding $p(x)$.

33. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$, find all numbers c for which the equation $A\vec{x} = cB\vec{x}$ has a nonzero solution. For each such c, find the corresponding $\vec{x}$.

34. Let $\vec{v}$ and $\vec{w}$ be eigenvectors of A with corresponding eigenvalues 2 and 3, respectively. Are $\vec{v}$ and $\vec{w}$ linearly dependent or linearly independent? Give a detailed explanation (or proof).

35. Check whether the following subset is a vector subspace: The set of all 2×2 matrices with determinant equal to 0.

36. (a) Find bases for the subspaces $\ker A$ and $\operatorname{im} A$ associated to the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 15 \\ 3 & 14 & 25 & -3 \end{bmatrix}.$$

(b) What is the orthogonal complement of the kernel of the above matrix A? Verify orthogonality.

37. Find an orthonormal basis for the subspace W of $\mathbb{R}^4$ spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

38. (a) Find the orthogonal projection of the vector $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ onto the image of the matrix $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 2 & 2 \end{bmatrix}$.

(b) Find a basis for $(\operatorname{im} A)^{\perp}$, i. e. the orthogonal complement of $\operatorname{im} A$ in $\mathbb{R}^3$.

39. (a) Compute the determinant of the matrix $\begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}$.

(b) Find all values of t for which the matrix A is invertible.

$$A = \begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix}.$$

40. (a) Compute the characteristic polynomial, the eigenvalues and corresponding eigenspaces of the matrix

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & -6 \\ 1 & 0 & -1 \end{bmatrix}.$$

(b) Is A diagonalizable? If not, explain why. If it is, write it as $A = PDP^{-1}$.

41. (a) Let A be a 3×3 matrix having the following properties:

i. $\ker A$ contains the vector $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$;

ii. $A^3\vec{v} = 8\vec{v}$, where $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$;

iii. multiplication by A leaves every vector on the line lying in the horizontal (x_1, x_2)-plane of equation $x_1 + x_2 = 0$ unchanged.

Find the eigenvalues and eigenvectors of A. Is A diagonalizable? Is A invertible?

(b) Find a basis for $\operatorname{im} A$.

42. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 4 & 6 \\ 1 & 4 & 6 \end{bmatrix}$.

(a) Find an orthonormal basis q_1, q_2, q_3 for the image of A.

(b) Find a vector q_4 such that q_1, q_2, q_3, q_4 is an orthonormal basis for $\mathbb{R}^4$.

43. Compute the following determinant:

$$\det \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ -1 & -6 & 1 & -1 \end{bmatrix}.$$

44. Let V be the plane in $\mathbb{R}^3$ defined by $x - 2y + z = 0$.

(a) Find a basis for V.

(b) Find a basis for the orthogonal complement $V^\perp$ of V.

(c) Find the matrix P of the projection onto V.

(d) Find all the eigenvalues and eigenvectors of P. *Hint: P is a projection matrix.*

45. Consider the matrix $A = \begin{bmatrix} \frac{4}{3} & -\frac{1}{3} \\ 1 & 0 \end{bmatrix}$.

(a) Diagonalize A, i. e. find an invertible matrix S and a diagonal matrix D such that $D = SAS^{-1}$.

(b) Calculate $A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

46. Consider

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 7 & 9 & 11 & 13 \\ 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(a) Is A invertible? Why or why not? Is $A^T A$ invertible? Why or why not?

(b) Write down a formula for the projection onto the subspace spanned by the columns of A. (Do not multiply out!)

47. Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 9 \\ 0 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

48. Prove which of the following are subspaces under matrix addition and which are not. For those that are subspaces, find a basis.

(a) The set of 3×3 matrices with determinant 1.

(b) The set of 2×2 orthogonal matrices.

(c) The set of 2×2 real skew-symmetric matrices. (Recall that A is skew-symmetric if $A^T = -A$.)

49. Let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Find an orthonormal basis for the space spanned by $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$. Find the projection of $\vec{v} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ onto this space.

50. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 5 & 2 \\ -3 & 0 \end{bmatrix}.$$

51. Two matrices A and B are similar if there is an invertible S such that $A = S^{-1}BS$.

(a) If A and B are similar matrices, prove that they have the same eigenvalues.

(b) Define $\operatorname{trace} A$, the trace of an $n \times n$ matrix, and define what it means for a matrix to be diagonalizable. Prove that if A is diagonalizable, then $\operatorname{trace}(A^k) = \lambda_1^k + \ldots + \lambda_n^k$, where the λ_i are the eigenvalues of A. *The result holds for all A, but you may assume that A is diagonalizable.*

52. Determine whether or not each of the following matrices is diagonalizable. In order to make your job easier, here are the characteristic polynomials.

(a) $A = \begin{bmatrix} 5 & 6 & -6 \\ 0 & -1 & 0 \\ 3 & 3 & -4 \end{bmatrix}$ $\qquad p_A(x) = (x-2)(x+1)^2$

(b) $B = \begin{bmatrix} 6 & 1 & 3 \\ -7 & -2 & -3 \\ -8 & -2 & -4 \end{bmatrix} \qquad p_B(x) = (x-2)(x+1)^2$

53. True or false?

    (a) If the columns of A are dependent, the $\det A$ must equal 0.

    (b) If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

    (c) Every linear transformation can be diagonalized.

54. Complete each of the following sentences with the *definition* of the italized word or phrase:

    (a) A linear transformation $T : V \to V$ is *diagonalizable* if ...

    (b) Let $T : V \to V$ be a linear transformation. A vector $\vec{v}$ is an *eigenvector* for T if it satisfies ...

    (c) A set of vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ is *orthonormal* if ...

55. Let A be an $n \times n$ $n$ilpotent matrix, that is for which there exists an integer k so that $A^k = 0$

    (a) Prove that 0 is the only eigenvalue of A.

    (b) Prove that the characteristic polynomial of A is x^n.

56. Let A and B be two $n \times n$ matrices with the property that $AB = BA$. Suppose that $\vec{v}$ is an eigenvector for A. Prove that $B\vec{v}$ is an eigenvector for A, provided it isn't the zero vector.

57. Let

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 0 & -3 & -7 \\ 0 & 0 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & 1 & 2 \\ 6 & 0 & 3 \\ 4 & -2 & -1 \end{bmatrix}.$$

Find $\det A$, $\det B$, $\det(A^{-4}B^2)$, $\det(B^T A^3 B^{-1})$ and $\det(A+B)$.

58. For any two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ show that $\vec{u} - \vec{v}$ is orthogonal to $\vec{u} + \vec{v}$ if and only if $\|\vec{u}\| = \|\vec{v}\|$.

59. Let A be a square matrix. True or false: if the columns of A are linearly independent, then so are the columns of A^{10}. Justify.

60. If V is the subspace spanned by $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, find an orthonormal basis for V and the projection matrix P onto V.

61. For the subspace V from the previous problem, find
    (a) a basis, not necessarily orthonormal, in the orthogonal complement $V^\perp$ of V;
    (b) the projection matrix Q onto $V^\perp$.

62. Are the matrices
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 6 & 7 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$
similar or not? Justify. *Hint: recall diagonalization.*

63. (a) Compute the determinant of the matrix
$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & -1 & 2 & 1 \\ 2 & 0 & -1 & 0 \\ 2 & -1 & 0 & -1 \end{bmatrix}.$$
    (b) Compute $\det(3A)$, $\det(-A)$, $\det B$ and $\det C$, where
$$B = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 6 & -1 & 2 & 1 \\ 6 & 0 & -1 & 0 \\ 6 & -1 & 0 & -1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 3 & -1 & 2 & 3 \\ 2 & -1 & 2 & 1 \\ 2 & 0 & -1 & 0 \\ 2 & -1 & 0 & -1 \end{bmatrix}.$$

Hint: how are B and C related to A?

64. Consider
$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 4 & 5 \\ 0 & 0 & 5 \end{bmatrix}.$$
Is the matrix A similar to a diagonal one? What about A^{-1}? Justify.

65. Diagonalize the matrix $A = \begin{bmatrix} 16 & -4 \\ -4 & 1 \end{bmatrix}$.

66. Consider the following pair of matrices:
$$A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}.$$
For each, determine the characteristic polynomial and the eigenvalues. Determine whether A, or B, or both are diagonalizable. If either of them is diagonalizable, find a matrix which diagonalizes it.

67. Let V be an n dimensional subspace of $\mathbb{R}^m$ and T an orthogonal transformation from V to V. Show that if $\vec{v} \in V$ and $T(\vec{v}) = \lambda\vec{v}$ with $\lambda \in \mathbb{R}$, then λ is either 1 or -1.

68. Suppose T and S are linear transformations from V to V which commute with one another. Suppose T has an eigenvalue λ such that the corresponding eigenspace W is one dimensional. Show that every nonzero element of W is an eigenvector of S.

69. Let
$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$
Compute explicitly the matrix A^{2006}. Find a formula for A^n in general.