# MATH 52/SPRING 2006/QUIZ

April 16, 2006

**Name:** SOLUTION KEY

There should be 8 pages in this packet, including this one. Check to make sure you have all of them. *Fully justify your answers!* No credit will be given if you do not show your work! And make sure you check your answers.

**No notes, no books, no calculators.**

| Number | Points | Score |
|---|---|---|
| 1 | 18 | |
| 2 | 18 | |
| 3 | 48 | |
| 4 | 16 | |
| **Total** | 100 | |
| Bonus | 5 | |

1. (18 points total) Let $\mathcal{A} : \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be the standard basis for $\mathbb{R}^2$. Let $\mathcal{B} : \vec{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ be another basis.

   (a) (6 points) Write down the change of basis matrix $S_{\mathcal{B}\to\mathcal{A}}$.

   (b) (6 points) Write down the change of basis matrix $S_{\mathcal{A}\to\mathcal{B}}$.

   (c) (6 points) Consider the vector $\vec{x} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$ in $\mathbb{R}^2$ with respect to the standard basis $\mathcal{A}$. Determine $[\vec{x}]_{\mathcal{B}}$.

a) $S_{\mathcal{B}\to\mathcal{A}} = \Big[ [\vec{v}_1]_{\mathcal{A}} \ [\vec{v}_2]_{\mathcal{A}} \Big] = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$

b) $S_{\mathcal{A}\to\mathcal{B}} = S_{\mathcal{B}\to\mathcal{A}}^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$

c) $[\vec{x}]_{\mathcal{B}} = S_{\mathcal{A}\to\mathcal{B}} [\vec{x}]_{\mathcal{A}} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

2. (20 points - 10 points each) Determine whether each set of vectors is linearly independent or linearly dependent. Justify your answers! *Hint: One of these does NOT require any calculations at all.*

(a) $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix};$

(b) $\vec{u}_1 = \begin{pmatrix} 3 \\ 8 \\ 11 \end{pmatrix}, \vec{u}_2 = \begin{pmatrix} -8 \\ 4 \\ 10 \end{pmatrix}, \vec{u}_3 = \begin{pmatrix} 7 \\ -2 \\ 18 \end{pmatrix}, \vec{u}_4 = \begin{pmatrix} 6 \\ -8 \\ 19 \end{pmatrix}.$

a) LINEARLY INDEPENDENT

1st method: Putting the vectors in a matrix, and row reducing, $\begin{pmatrix} 1 & 2 & 6 \\ 0 & 1 & 0 \\ 2 & 4 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we see the matrix has full rank, so its columns are linearly independent.

2nd method: $\{\vec{v}_1, \vec{v}_3\}$ are linearly independent, since one is not a multiple of another. And furthermore, $\vec{v}_2$ is not a linear combination of $\vec{v}_1$ and $\vec{v}_3$ since $\vec{v}_1$ and $\vec{v}_3$ have 0's in 2nd position and $\vec{v}_2$ does not. So $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly independent.

b) LINEARLY DEPENDENT

1st method: If we form a matrix $A$ of these vectors, it will be $3 \times 4$ and hence $\text{rank}(A) < 4$. So the columns cannot be independent.

2nd method: These vectors are in $\mathbb{R}^3$. Since $\mathbb{R}^3$ has dimension 3, any set of vectors with <u>more</u> than three elements is <u>dependent</u>.

3. (48 points total) Let $B = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$ and $T$ be the linear transformation from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$ defined by $T(M) = BM$. Consider the basis for $\mathbb{R}^{2\times 2}$ consisting of

$$E_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ E_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},\ E_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ E_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The $4 \times 4$ matrix $A = \begin{bmatrix} 3 & & & 0 \\ 3 & & & 0 \\ 0 & & & 3 \\ 0 & & & 3 \end{bmatrix}$ represents $T$ with respect to the basis $\mathcal{B} : E_1, \ldots, E_4$.

(a) (14 points) Supply the entries in the 2nd and 3rd columns of $A$.

<u>2nd column</u>

$$T(E_2) = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} = 3E_1 + 3E_2$$

So $$[T(E_2)]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 3 \\ 0 \\ 0 \end{pmatrix}\begin{array}{l} \leftarrow E_1 \\ \leftarrow E_2 \\ \leftarrow E_3 \\ \leftarrow E_4 \end{array}$$

<u>3rd column</u>

$$T(E_3) = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 0 & 3 \end{pmatrix} = 3E_3 + 3E_4$$

So $$[T(E_3)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 3 \end{pmatrix}$$

So $$A = \begin{bmatrix} 3 & 3 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

(b) (18 points) Find bases for the kernel and the image of the matrix $A$ from part (a).

Row Reduce $A$:

$$\begin{pmatrix} 3 & 3 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rightsquigarrow \begin{array}{c} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{matrix} L & F & L & F \end{matrix} \end{array}$$

So $\ker(A) = \left\{ \begin{pmatrix} -s \\ s \\ -t \\ t \end{pmatrix} \right\} = \text{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$

and $\ker(A)$ has basis $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

Further, image has as basis all "leading columns",

i.e. $\begin{pmatrix} 3 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \\ 3 \end{pmatrix}$.

(c) (8 points) Write down bases for the kernel and the image of the transformation $T$ defined in part (a).

Since $A = [T]_{\mathcal{B}}$ has kernel & image as given in part (b), we have

basis for $\ker(T)$: $-E_1 + E_2 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$ ← from $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$-E_3 + E_4 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ ← from $\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

$\boxed{\text{basis is } \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}}$

basis for $\operatorname{im}(T)$: $3E_1 + 3E_2 = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$ ← from $\begin{pmatrix} 3 \\ 3 \\ 0 \\ 0 \end{pmatrix}$

$3E_3 + 3E_4 = \begin{pmatrix} 0 & 3 \\ 0 & 3 \end{pmatrix}$ ← from $\begin{pmatrix} 0 \\ 0 \\ 3 \\ 3 \end{pmatrix}$

$\boxed{\text{basis is } \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 0 & 3 \end{pmatrix}}$

(d) (8 points) Verify the Rank-Nullity Theorem for $T$. (That is, write down the equation given in the Rank-Nullity Theorem, and calculate each of the three values it relates. Then check that these three values satisfy the equation.)

$$\dim(\mathbb{R}^{2\times 2}) = \operatorname{rank}(T) + \operatorname{nullity}(T)$$

We have $\dim(\mathbb{R}^{2\times 2}) = 4$

$\left.\begin{array}{l} \operatorname{rank}(T) = 2 \\ \operatorname{nullity}(T) = 2 \end{array}\right\}$ from part (c)

And $2 + 2 = 4$ ✓

4. (14 points) Consider the following subset of $P_2$.

$$S = \{f(t) \in P_2 : f'(4) = 0\}$$

Is this a subspace of $P_2$? Justify your answer.

① The zero polynomial $f(t)=0$ satisfies $f'(4)=0$,
so $0 \in S$.

② Suppose $f(t), g(t) \in S$. Then $f'(4)=0$ and
$g'(4)=0$.
Then $(f+g)(t) = f(t)+g(t)$ satisfies
$(f+g)'(4) = f'(4)+g'(4) = 0+0=0$
So $f(t)+g(t) \in S$. So $S$ is closed under addition.

③ Suppose $f(t) \in S$ and $k \in \mathbb{R}$. Then
$kf(t)$ satisfies $kf'(4) = k \cdot 0 = 0$
So $kf(t) \in S$. So $S$ is closed
under scalar multiplication.

Since $S$ contains $0$ and is closed under scalar multiplication & addition, it is a subspace.

Alternate method: $S = \{f(t) \in P_2 : f'(4)=0\} = \{a+bt+ct^2 : b+2c(4)=0\}$
$= \{a+(-8c)t+ct^2\}$
$= \text{span}\{1, -8t+t^2\}$

Therefore, since $S$ is a span, and spans are subspaces, $S$ is a subspace.

5. BONUS PROBLEM (5 points) Is the following linear transformation an isomorphism? Justify your answer. If it is an isomorphism, find its inverse. If it is not an isomorphism, determine the rank and nullity of the transformation.

   $T : \mathbb{R}^{2\times 2} \to \mathbb{R}$ given by $T(M) = \det(M)$.

This was an (unintentional – oops!) trick question:

T as given is <u>not</u> a linear transformation.