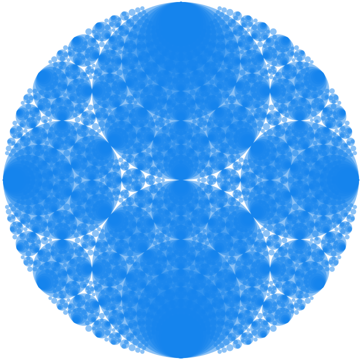
An illustration in number theory

Katherine E. Stange



AMS Joint Central/Western Sectional Meeting, March 22, 2019

Let us begin with a sandpile…

an Abelian sandpile

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	16	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

an Abelian sandpile

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	1	0	0	0
0	0	1	12	1	0	0
0	0	0	1	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

an Abelian sandpile

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	2	0	0	0
0	0	2	8	2	0	0
0	0	0	2	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

an Abelian sandpile

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	3	0	0	0
0	0	3	4	3	0	0
0	0	0	3	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

an Abelian sandpile

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	4	0	0	0
0	0	4	0	4	0	0
0	0	0	4	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

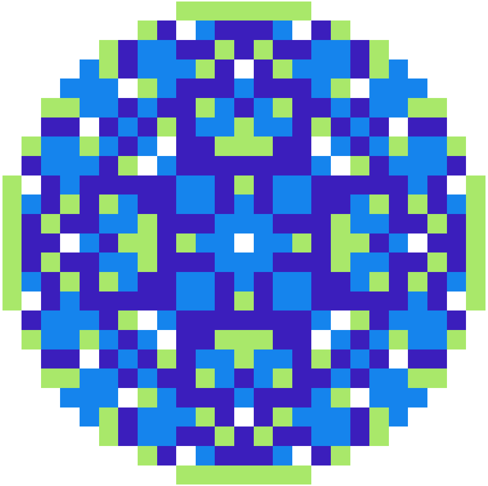
an Abelian sandpile

0	0	0	0	0	0	0
0	0	0	1	0	0	0
0	0	2	0	2	0	0
0	1	0	4	0	1	0
0	0	2	0	2	0	0
0	0	0	1	0	0	0
0	0	0	0	0	0	0

an Abelian sandpile

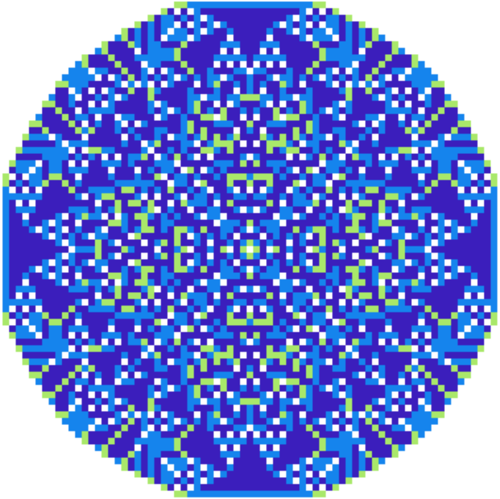
0	0	0	0	0	0	0
0	0	0	1	0	0	0
0	0	2	1	2	0	0
0	1	1	0	1	1	0
0	0	2	1	2	0	0
0	0	0	1	0	0	0
0	0	0	0	0	0	0

an Abelian sandpile



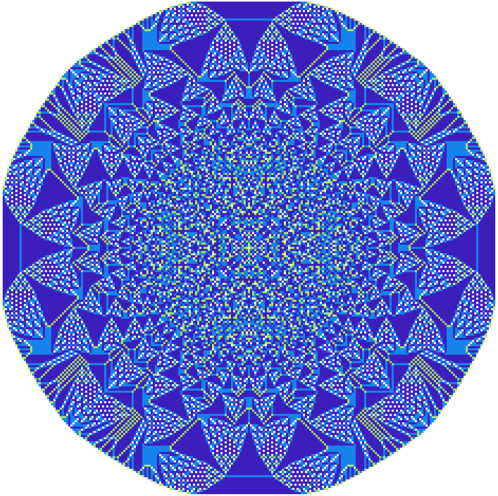
1,000 grains

an Abelian sandpile



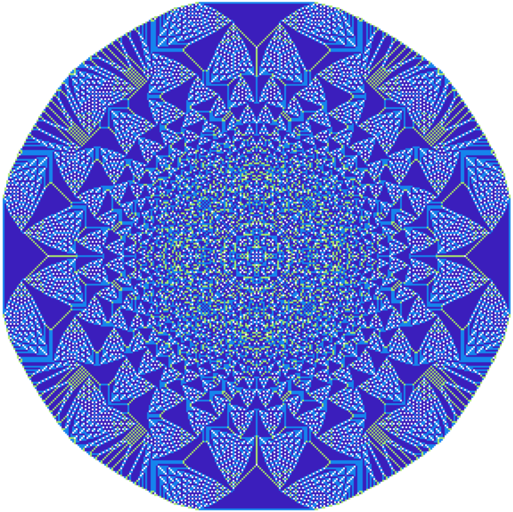
10,000 grains

an Abelian sandpile



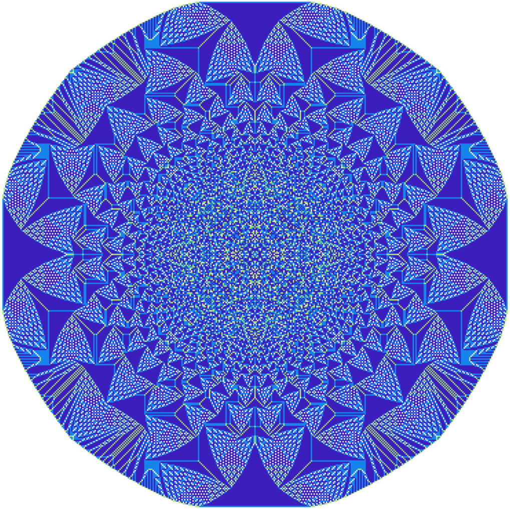
100,000 grains

an Abelian sandpile



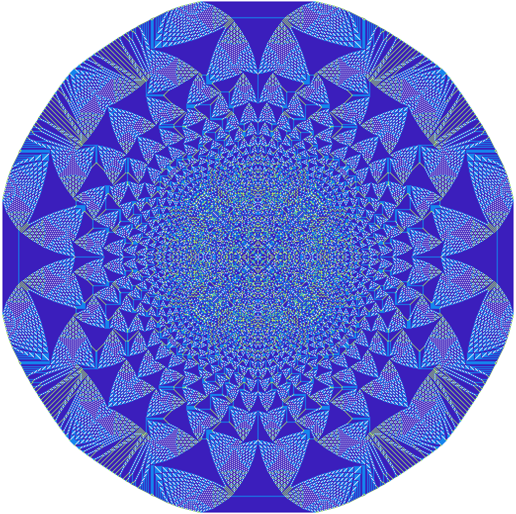
150,000 grains

an Abelian sandpile



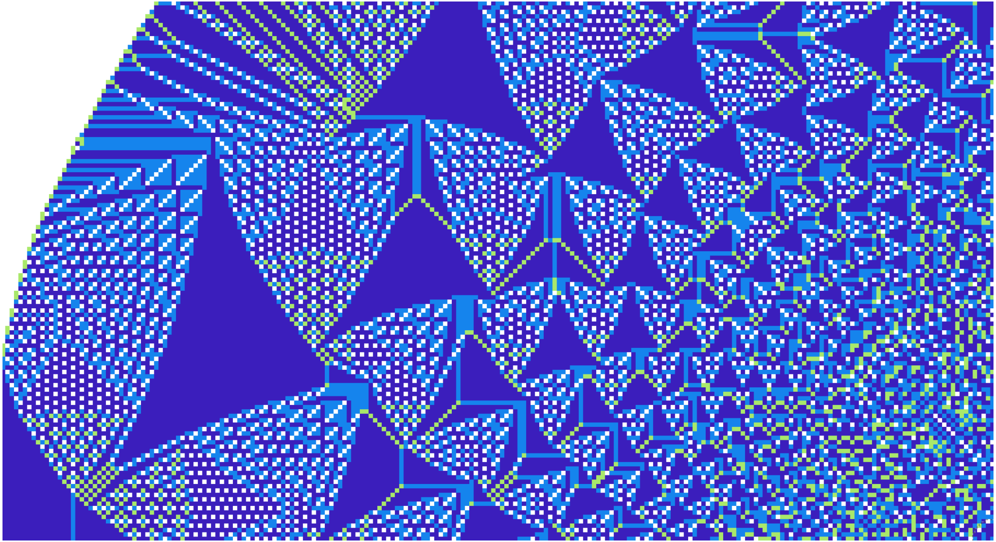
250,000 grains

an Abelian sandpile



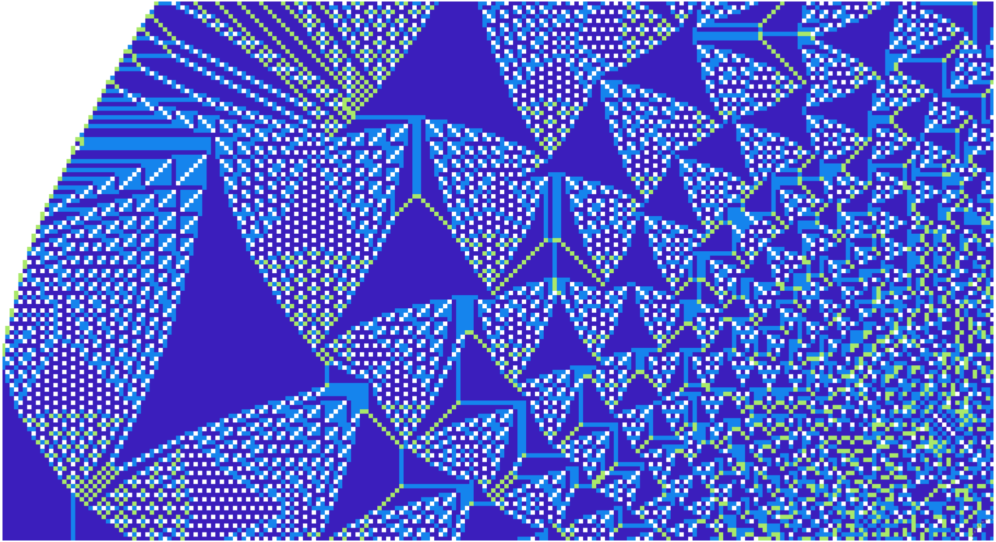
500,000 grains

an Abelian sandpile



detail

an Abelian sandpile



Bak, Tang, Wiesenfeld: self-organized criticality

the discrete Laplacian

acting on functions $g : \mathbb{Z}^2 \to \mathbb{Z}$

$$\Delta g(x) = \sum_{y \sim x} (g(y) - g(x)).$$

$$\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\quad \mapsto \quad
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -4 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

odometer　　　　sandpile

the discrete Laplacian

acting on functions $g : \mathbb{Z}^2 \to \mathbb{Z}$

$$\Delta g(x) = \sum_{y \sim x} (g(y) - g(x)).$$

$$\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\quad \mapsto \quad
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -4 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

bounded odometer

sandpile stabilizes to 0 in finite time

stablizable

2	2	2	2	1	2	2
2	2	2	1	6	1	2
2	2	2	2	1	2	2
2	2	1	2	2	2	2
2	1	6	1	2	2	2
2	2	1	2	2	2	2
2	2	2	2	2	2	2

stabilizable

2	2	2	2	1	2	2
2	2	2	1	6	1	2
2	2	2	2	1	2	2
2	2	1	2	2	2	2
2	1	6	1	2	2	2
2	2	1	2	2	2	2
2	2	2	2	2	2	2

stabilizable

2	2	2	2	2	2	2
2	2	2	2	2	2	2
2	2	2	2	2	2	2
2	2	2	2	2	2	2
2	2	2	2	2	2	2
2	2	2	2	2	2	2
2	2	2	2	2	2	2

unstabilizable

4	0	4	0	4	0	4
0	4	0	4	0	4	0
4	0	4	0	4	0	4
0	4	0	4	0	4	0
4	0	4	0	4	0	4
0	4	0	4	0	4	0
4	0	4	0	4	0	4

unstabilizable

4	0	4	0	4	0	4
0	4	0	4	0	4	0
4	0	4	0	4	0	4
0	4	0	4	0	4	0
4	0	4	0	4	0	4
0	4	0	4	0	4	0
4	0	4	0	4	0	4

unstabilizable

4	0	4	1	0	1	4
0	4	0	4	2	4	0
4	0	4	1	0	1	4
0	4	0	4	2	4	0
4	0	4	1	0	1	4
0	4	0	4	2	4	0
4	0	4	1	0	1	4

unstabilizable

4	0	4	1	0	1	4
0	4	0	$\boxed{4}$	2	$\boxed{4}$	0
4	0	4	1	0	1	4
0	4	0	$\boxed{4}$	2	$\boxed{4}$	0
4	0	4	1	0	1	4
0	4	0	$\boxed{4}$	2	$\boxed{4}$	0
4	0	4	1	0	1	4

unstabilizable

4	0	4	3	0	3	4
0	4	1	0	4	0	1
4	0	4	3	0	3	4
0	4	1	0	4	0	1
4	0	4	3	0	3	4
0	4	1	0	4	0	1
4	0	4	3	0	3	4

unstabilizable

4	0	$\boxed{4}$	3	0	3	$\boxed{4}$
0	4	1	0	4	0	1
4	0	$\boxed{4}$	3	0	3	$\boxed{4}$
0	4	1	0	4	0	1
4	0	$\boxed{4}$	3	0	3	$\boxed{4}$
0	4	1	0	4	0	1
4	0	$\boxed{4}$	3	0	3	$\boxed{4}$

unstabilizable

4	1	0	4	0	4	0
0	4	3	0	4	0	3
4	1	0	4	0	4	0
0	4	3	0	4	0	3
4	1	0	4	0	4	0
0	4	3	0	4	0	3
4	1	0	4	0	4	0

unstabilizable

4	1	0	4	0	4	0
0	$\boxed{4}$	3	0	4	0	3
4	1	0	4	0	4	0
0	$\boxed{4}$	3	0	4	0	3
4	1	0	4	0	4	0
0	$\boxed{4}$	3	0	4	0	3
4	1	0	4	0	4	0

unstabilizable

4	3	0	4	0	4	0
1	0	4	0	4	0	4
4	3	0	4	0	4	0
1	0	4	0	4	0	4
4	3	0	4	0	4	0
1	0	4	0	4	0	4
4	3	0	4	0	4	0

unstabilizable

4	3	0	4	0	4	0
1	0	4	0	4	0	4
4	3	0	4	0	4	0
1	0	4	0	4	0	4
4	3	0	4	0	4	0
1	0	4	0	4	0	4
4	3	0	4	0	4	0

unstabilizable

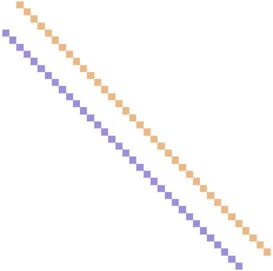
0	4	0	4	0	4	0
3	0	4	0	4	0	4
0	4	0	4	0	4	0
3	0	4	0	4	0	4
0	4	0	4	0	4	0
3	0	4	0	4	0	4
0	4	0	4	0	4	0

unstabilizable

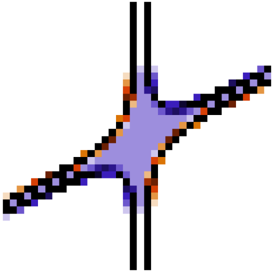
0	4	0	4	0	4	0
4	0	4	0	4	0	4
0	4	0	4	0	4	0
4	0	4	0	4	0	4
0	4	0	4	0	4	0
4	0	4	0	4	0	4
0	4	0	4	0	4	0

Laplacians of integral odometers

odometer $\mapsto$ sandpile



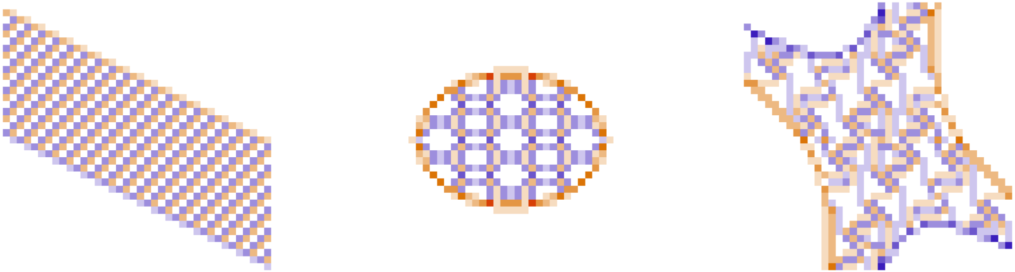
$ax + by + c \mapsto 0$



$ax^2 + by^2 + cxy \mapsto 2a + 2b$

Laplacians of rational odometers

odometer $\mapsto$ sandpile



$$\lfloor ax+by+c \rfloor \mapsto \left\{ \begin{matrix} \text{periodic} \\ \text{avg} = 0 \end{matrix} \right\}$$

$$\lfloor ax^2+by^2+cxy \rfloor \mapsto \left\{ \begin{matrix} \text{periodic} \\ \text{avg} = 2a+2b \end{matrix} \right\}$$

superharmonic quadratic growths

Levine, Pegden, Smart

Let $A \in \mathrm{Sym}_{2\times 2}(\mathbb{R})$. We say an odometer g has *quadratic growth* A if

$$g(x) = x^t A x + o(|x|^2)$$

and we say that A is *superharmonic* if there is an odometer g with quadratic growth A and

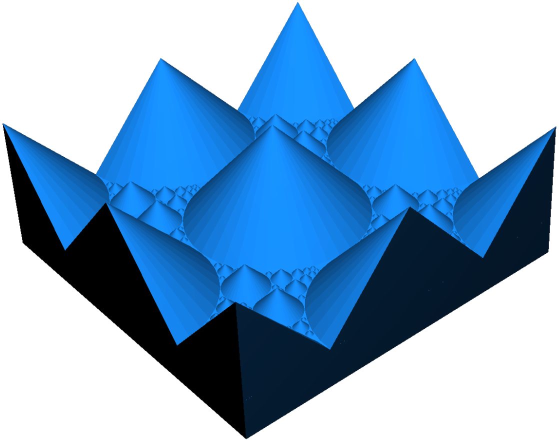
$$\Delta g \leq 1.$$

Let Γ be the set of such superharmonic A, as a subset of $\mathbb{R}^3$:

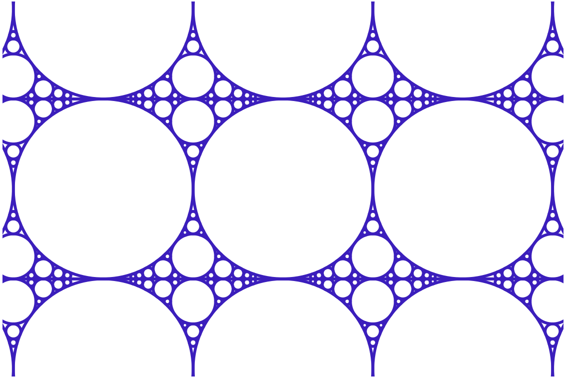
$$A = \frac{1}{2}\begin{pmatrix} t-x & y \\ y & t+x \end{pmatrix} \quad \mapsto \quad (x, y, t)$$

Then $t = \mathrm{tr}(A)$.

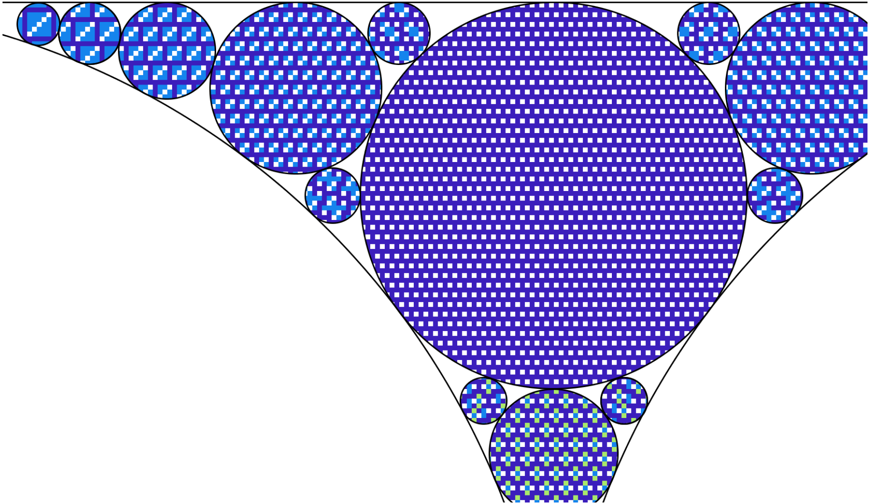
computer calculation of Γ



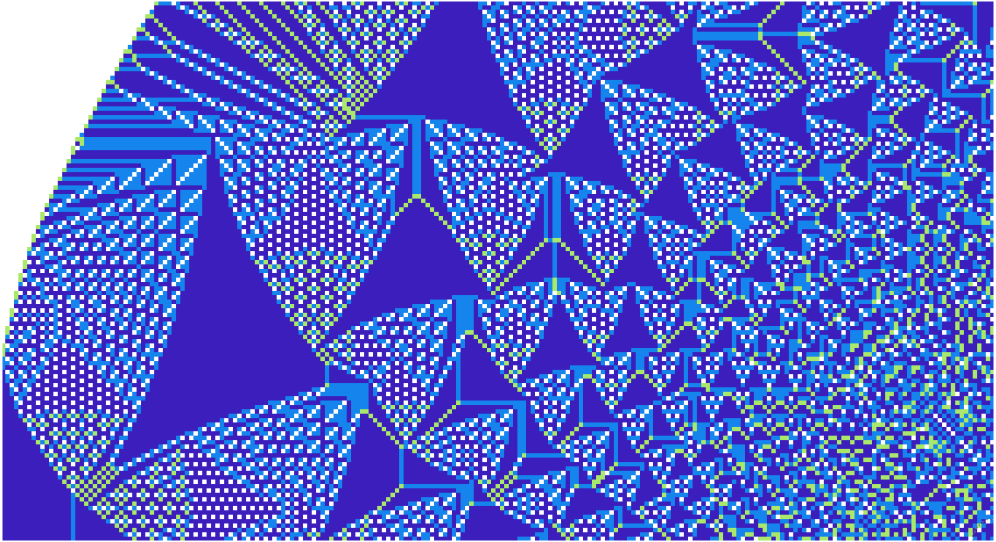
computer calculation of Γ



sandpiles of Γ peaks

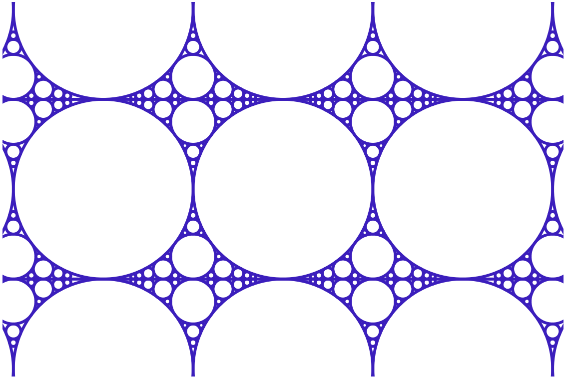


detail of sandpile

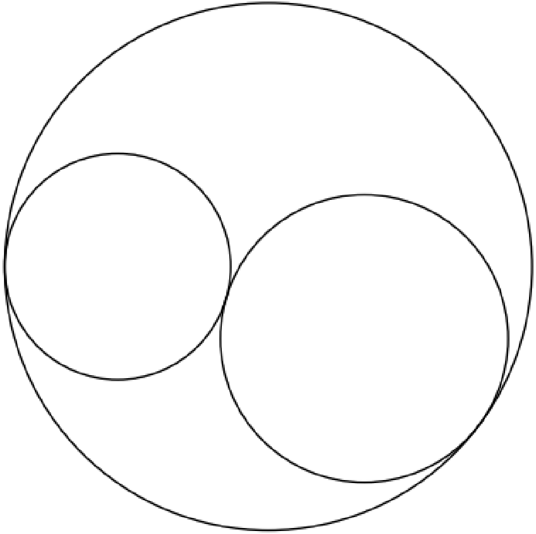


detail

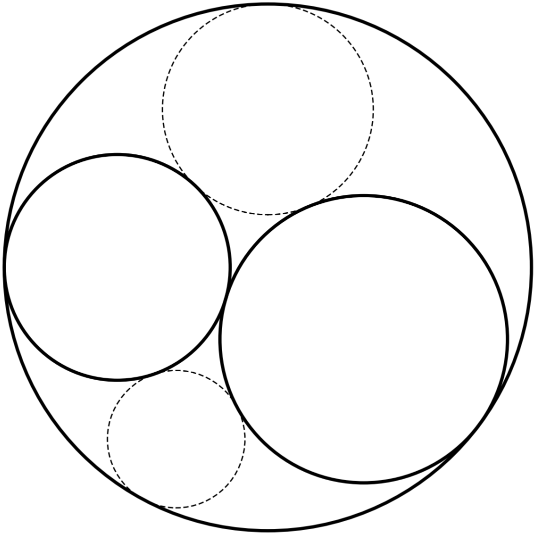
computer calculation of Γ



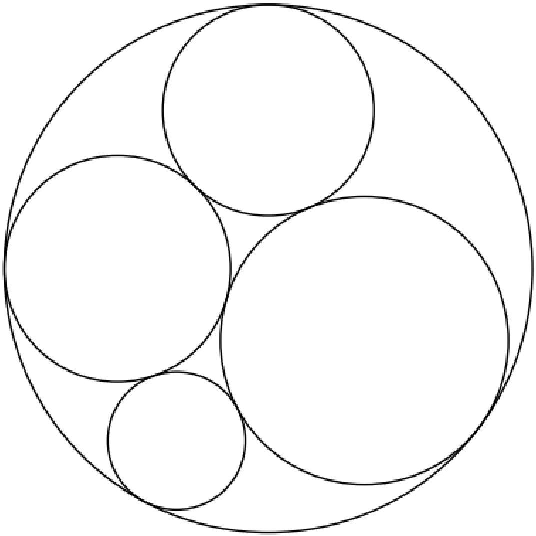
Apollonian circle packing



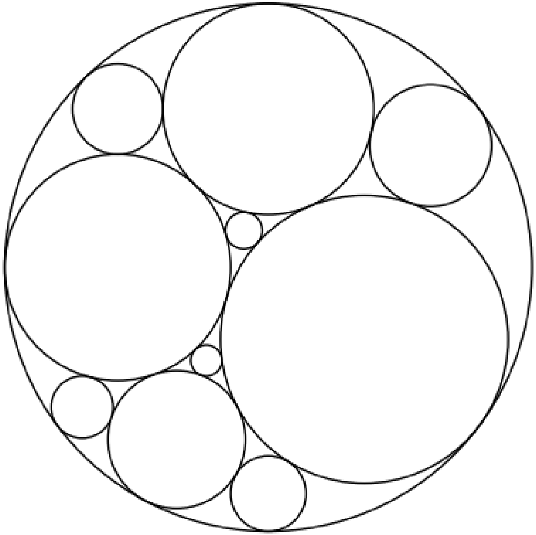
Apollonian circle packing



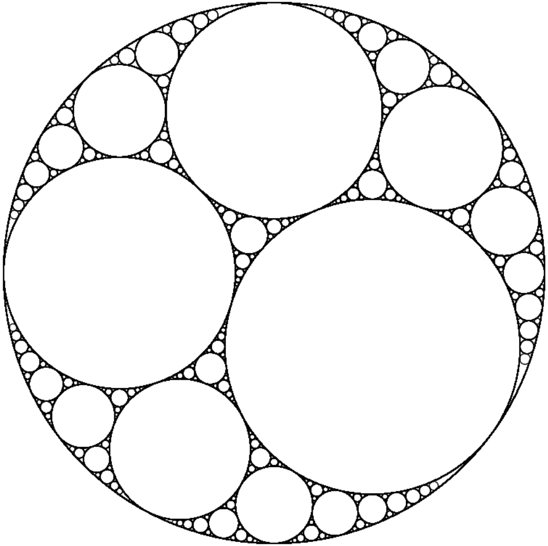
Apollonian circle packing



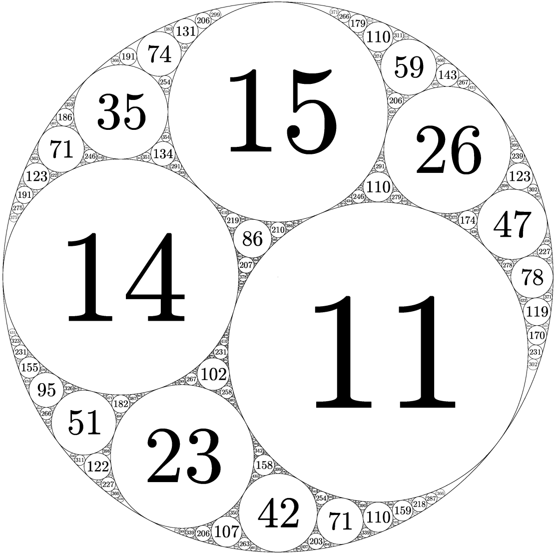
Apollonian circle packing



Apollonian circle packing



Apollonian circle packing



integer curvatures

Curvatures a, b, c, d of four mutually tangent circles (a *Descartes quadruple*) satisfy

$$2(a^2+b^2+c^2+d^2)=(a+b+c+d)^2.$$

Given a, b, c, there are two possibilities d and d' satisfying

$$d+d'=2(a+b+c).$$

Therefore

$$a,b,c,d\in\mathbb{Z} \quad\Longrightarrow\quad \text{everything}\in\mathbb{Z}.$$

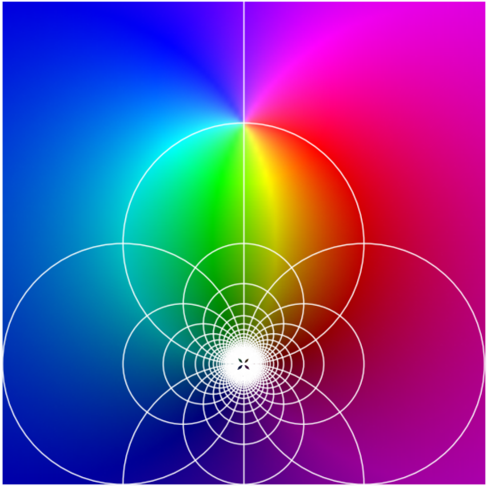
local-to-global

Conjecture (Graham-Lagarias-Mallows-Wilks-Yan, Fuchs-Sanden):

In a primitive integral Apollonian circle packing, curvatures satisfy a congruence condition modulo 24, *and all sufficiently large integers satisfying this condition appear.*

Bourgain-Fuchs: curvatures have positive density
Bourgain-Kontorovich: density one occur

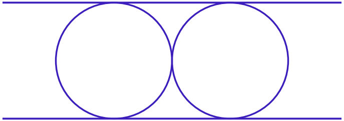
the language for circles: Möbius transformations



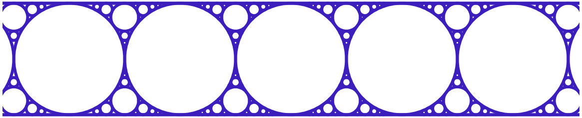
$\mathrm{PSL}_2(\mathbb{C})$ acts on the extended complex plane, taking circle to circles:

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \quad \mapsto \quad \left(z \mapsto \frac{\alpha z + \gamma}{\beta z + \delta} \right)$$

the Apollonian group in $\mathrm{PSL}_2(\mathbb{Z}[i])$

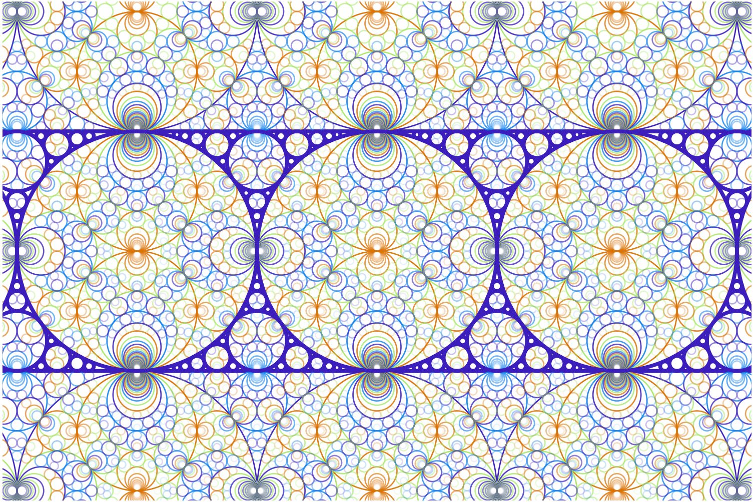


Base quadruple: $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \begin{pmatrix}-1 & i\\ 0 & 1\end{pmatrix}, \quad \begin{pmatrix}0 & -1\\ -1 & i-1\end{pmatrix}, \quad \begin{pmatrix}1 & 1-i\\ 1 & -i\end{pmatrix}.$



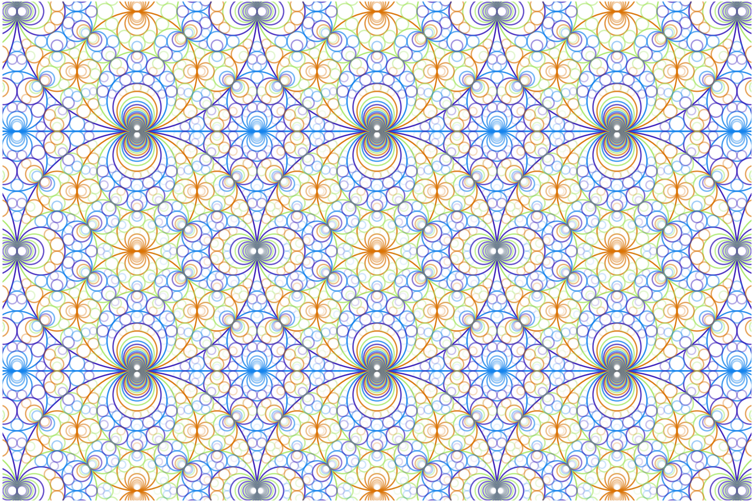
Orbit under Apollonian group: $\mathscr{A} = \left\langle \begin{pmatrix}-1 & 2\\ -1 & 1\end{pmatrix}, \begin{pmatrix}1 & -1\\ 2 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \begin{pmatrix}1 & 1-i\\ 0 & -1\end{pmatrix} \right\rangle.$

Schmidt arrangement of $\mathbb{Q}(i)$



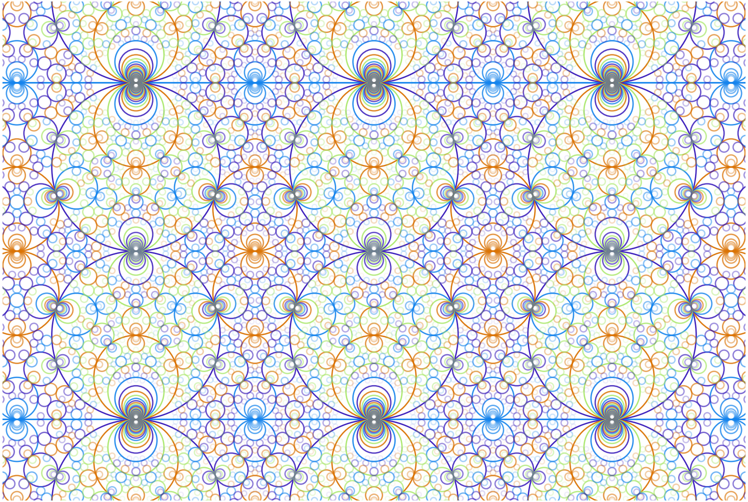
orbit of real line under $\mathrm{PSL}_2(\mathbb{Z}[i])$

Schmidt arrangement of $\mathbb{Q}(i)$



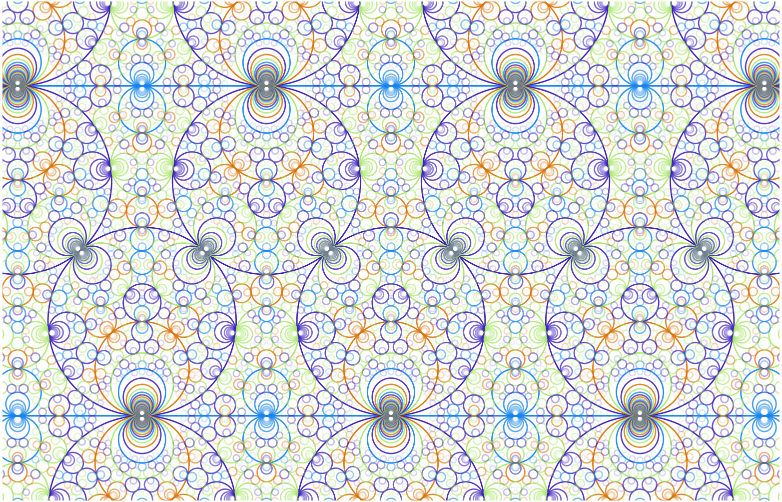
orbit of real line under $\mathrm{PSL}_2(\mathbb{Z}[i])$

Schmidt arrangement of $\mathbb{Q}(\sqrt{-2})$



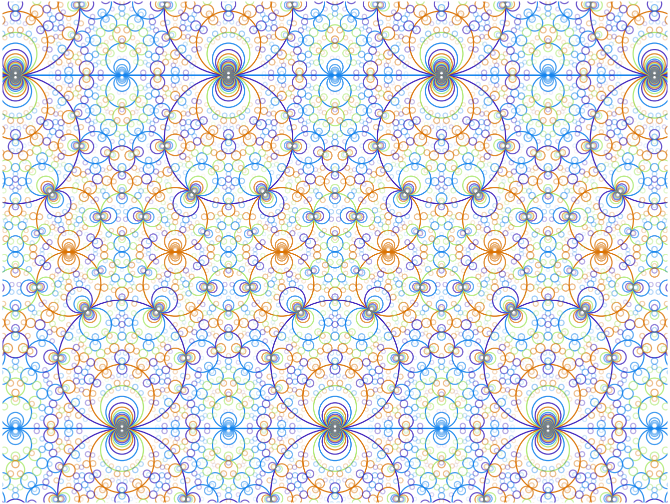
orbit of real line under $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-2}])$

Schmidt arrangement of $\mathbb{Q}(\sqrt{-7})$



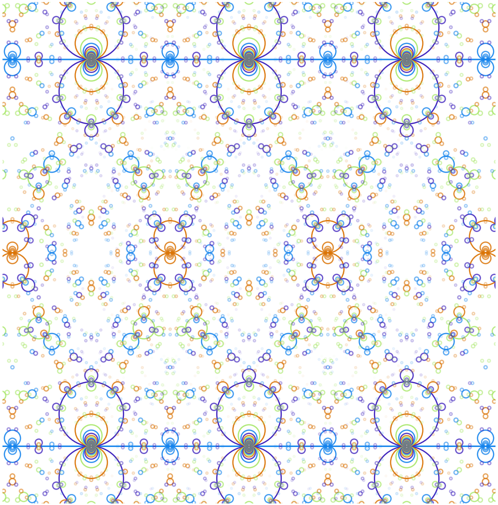
orbit of real line under $\mathrm{PSL}_2(\mathbb{Z}[\frac{1+\sqrt{-7}}{2}])$

Schmidt arrangement of $\mathbb{Q}(\sqrt{-11})$



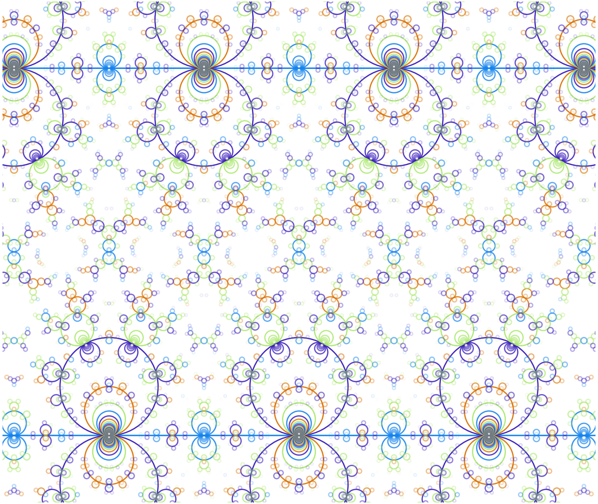
orbit of real line under $\mathrm{PSL}_2(\mathbb{Z}[\frac{1+\sqrt{-11}}{2}])$

Schmidt arrangement of $\mathbb{Q}(\sqrt{-6})$



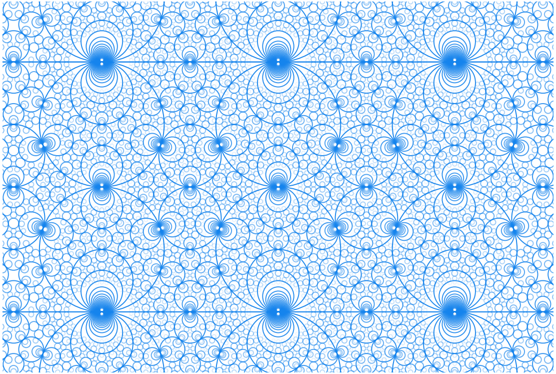
orbit of real line under $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-6}])$

Schmidt arrangement of $\mathbb{Q}(\sqrt{-15})$



orbit of real line under $\mathrm{PSL}_2(\mathbb{Z}[\frac{1+\sqrt{-15}}{2}])$

Illustrating the Arithmetic of Imaginary Quadratic Fields

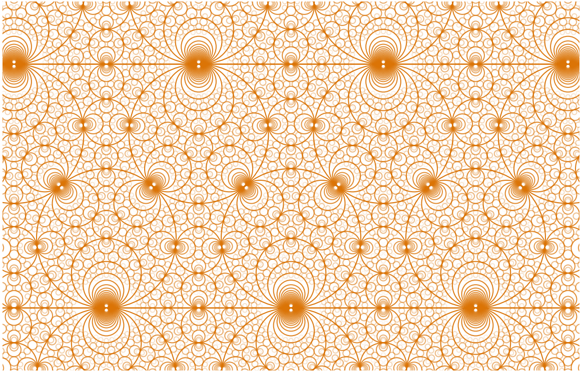


tangency points = rational points of trivial class

$\alpha/\beta \in K$ such that (α, β) is principal

size of the pencil $= 1/N(\beta)$

Illustrating the Arithmetic of Imaginary Quadratic Fields



circles = ideal classes of orders
(which are trivial when extended to maximal ideal)

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \longleftrightarrow \beta\mathbb{Z} + \delta\mathbb{Z}$$

curvature of circle = conductor of the order

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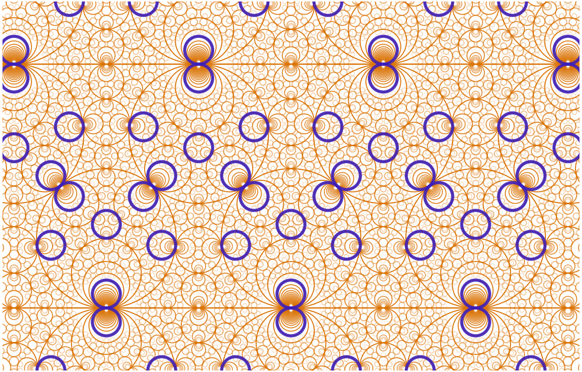

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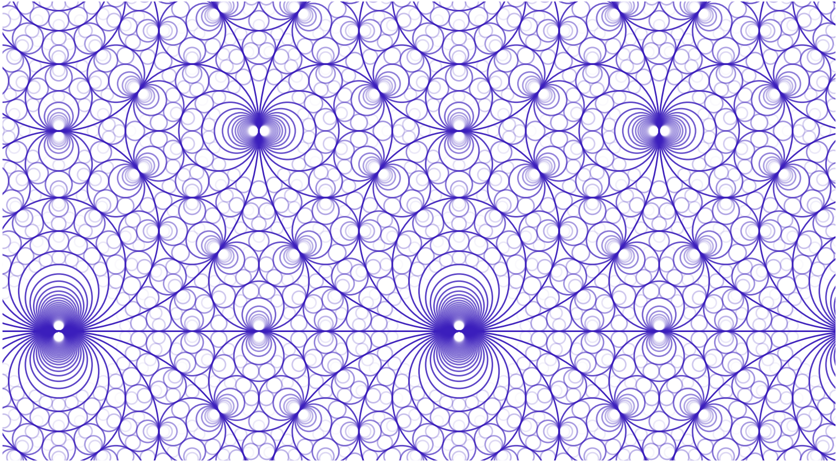
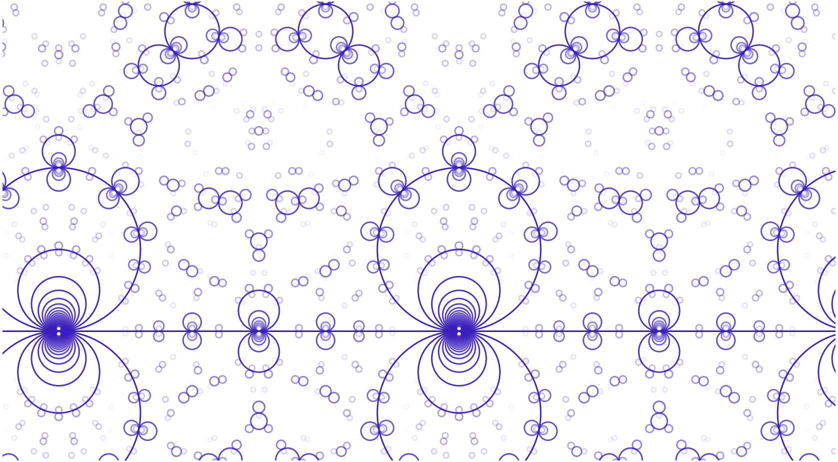


Euclideanity = tangency (or topological) connectedness

Illustrating the Arithmetic of Imaginary Quadratic Fields



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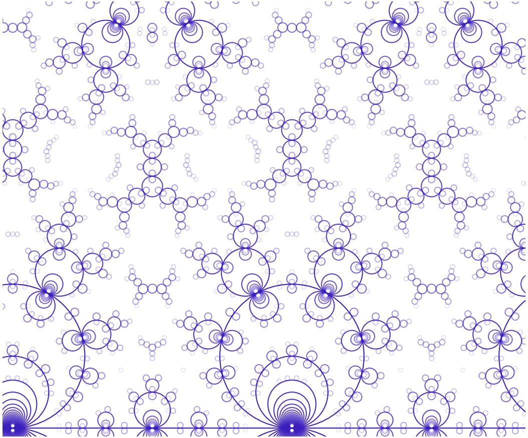
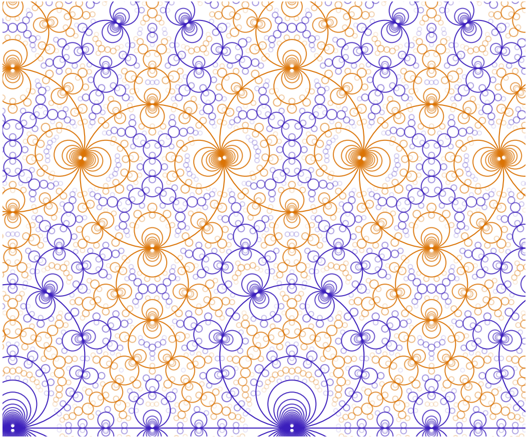


2-part of the class group = maximal discrete extension of $\mathrm{PSL}_2(\mathcal{O}_K)$

Illustrating the Arithmetic of Imaginary Quadratic Fields



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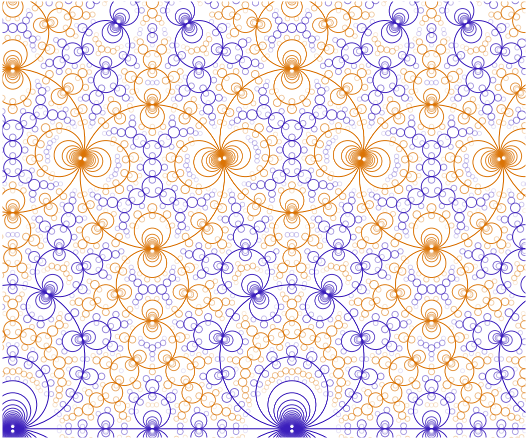
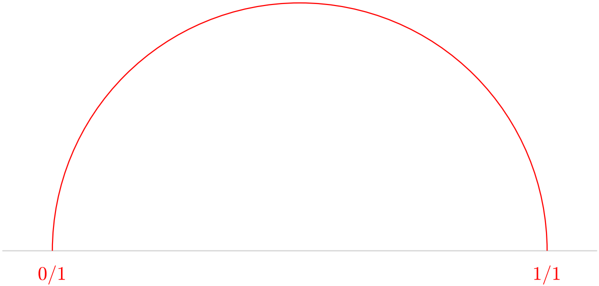


Daniel Martin: other parts of the class group!

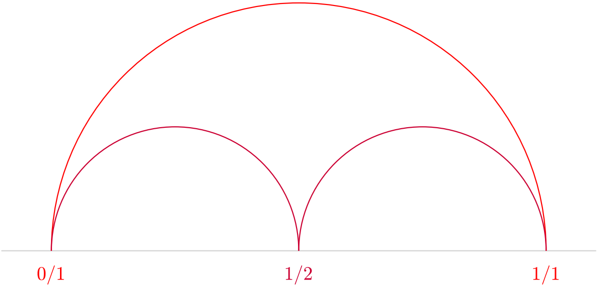
AMS Meeting, 4pm on Saturday, Emerging Connections with Number Theory

the Farey subdivision



$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

the Farey subdivision



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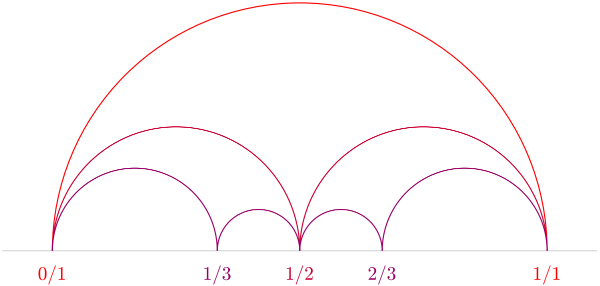

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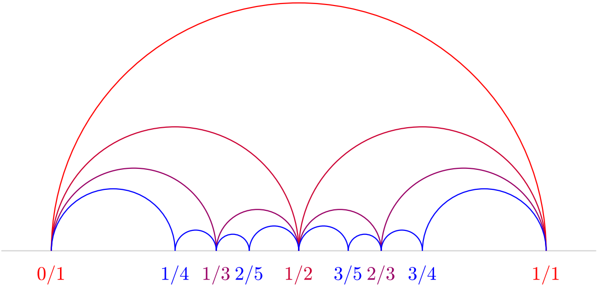
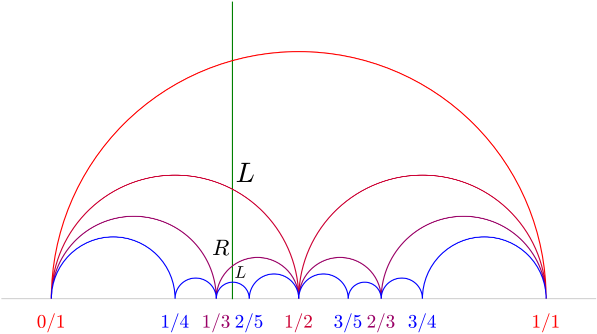


$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

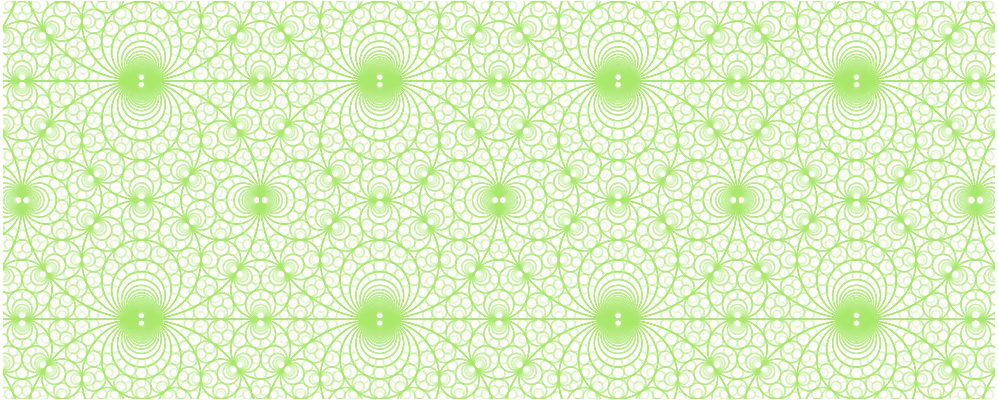
continued Fractions: $\mathbb{Q}$ in $\mathbb{R}$



$LRL\cdots$ is the continued fraction expansion
endpoints of pierced bubbles are good approximations:

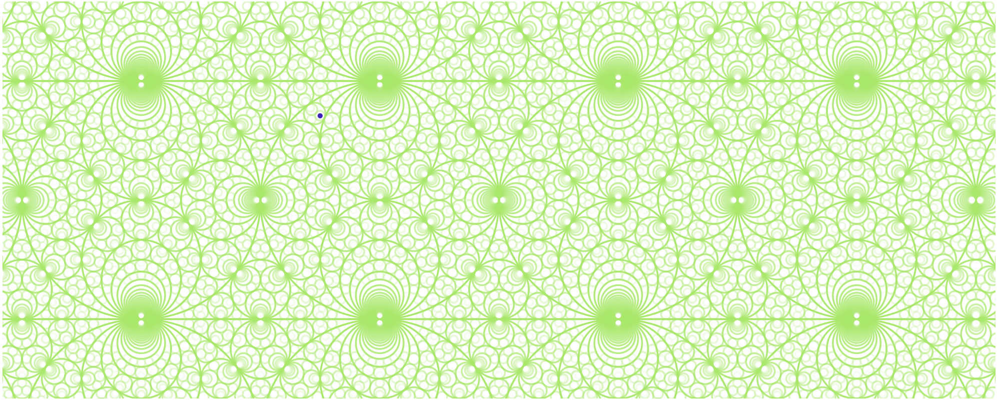
$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

continued fractions: $\mathbb{Q}(i)$ in $\mathbb{C}$



Asmus Schmidt, 1975: continued fractions by zooming in through the regions tangency points of the 'moves' are good approximations

continued fractions: $\mathbb{Q}(i)$ in $\mathbb{C}$



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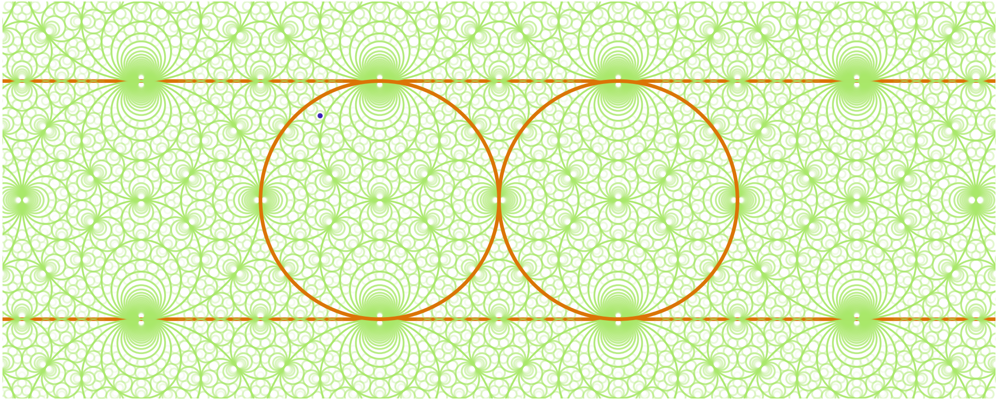

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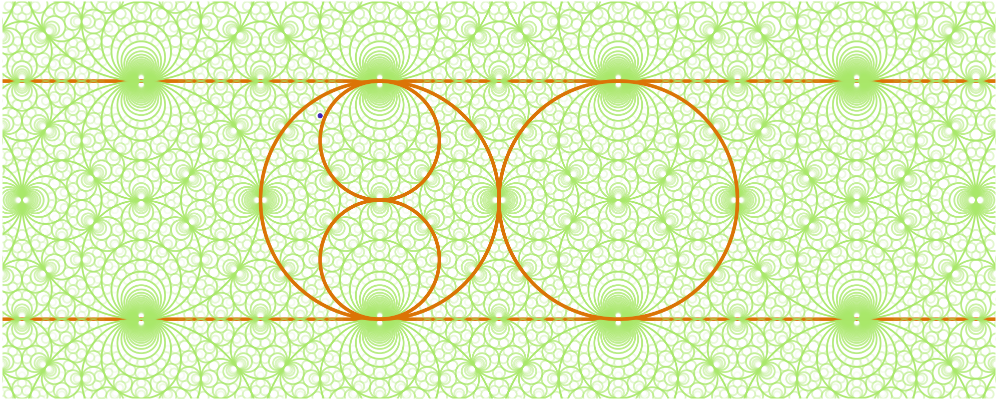

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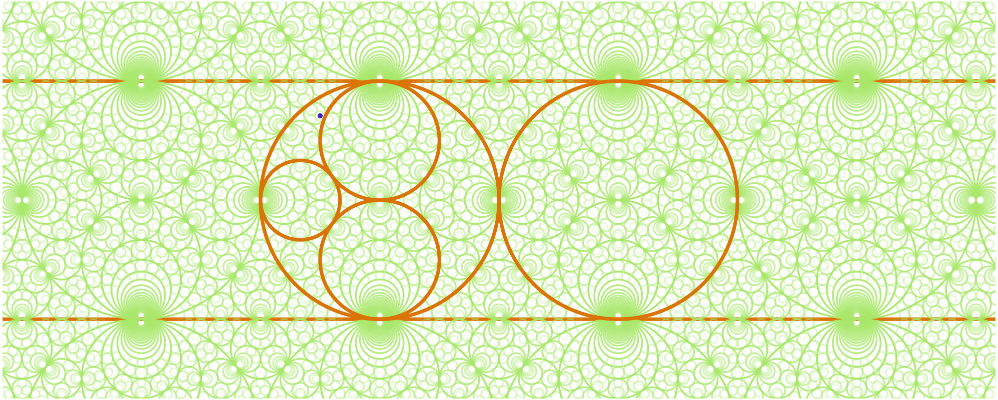

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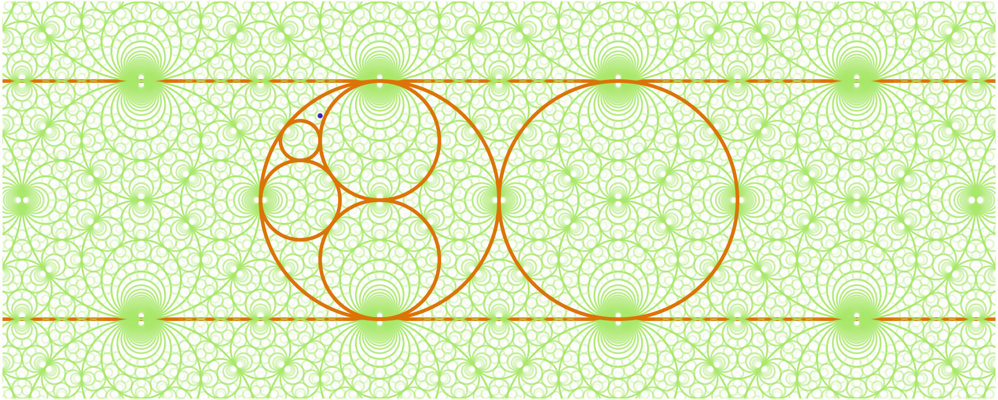

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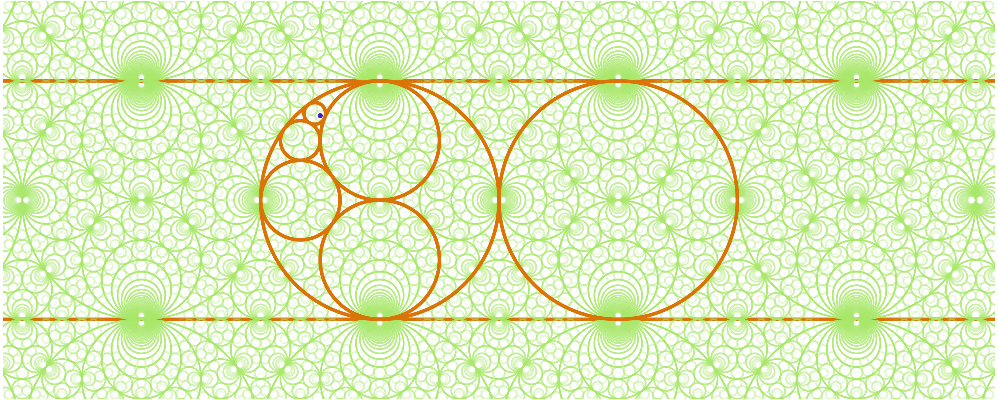

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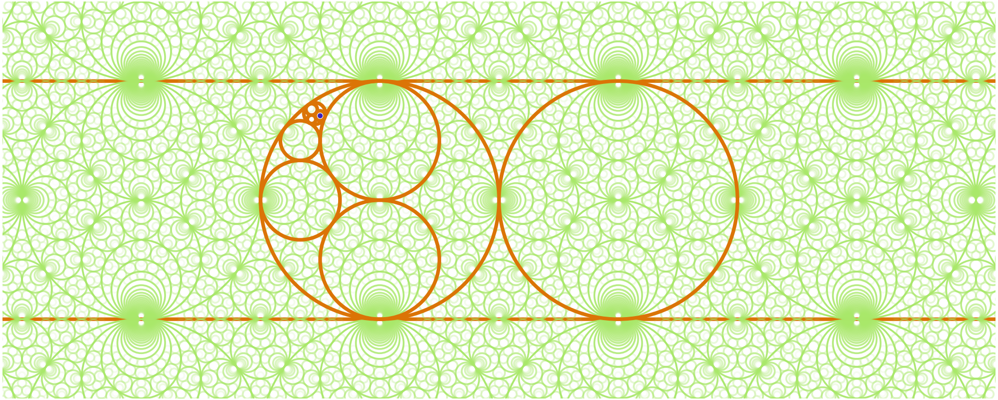

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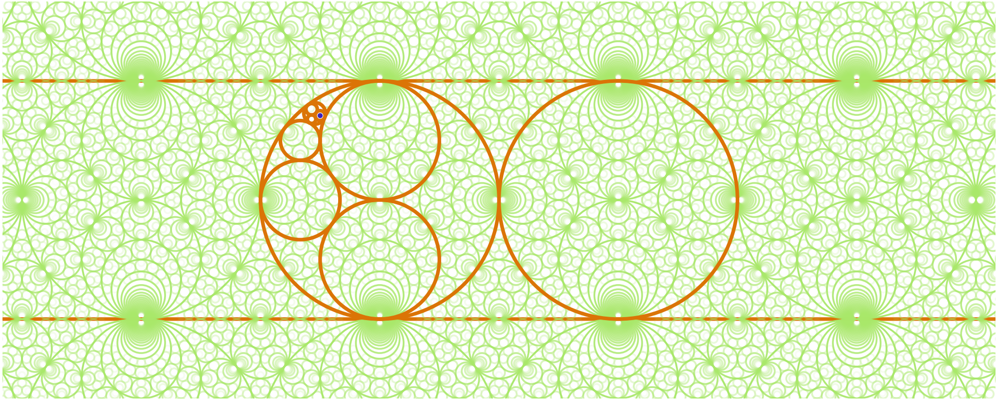

Daniel Martin: continued fractions even when non-Euclidean!

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lattices and circles from $\mathrm{PSL}_2(\mathscr{O}_K)$

Λ_M
lattice
$\beta\mathbb{Z}+\delta\mathbb{Z}$

M
Möbius transformation
$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$

C_M
circle
$M \cdot \widehat{\mathbb{R}}$

lattices and circles from quadratic odometers

$$
\begin{tikzcd}
& \substack{\Lambda_A \\ \text{lattice} \\ \{\mathbf{v} \in \mathbb{Z}^2 : A\mathbf{v} \in \mathbb{Z}^2\}} \\
\substack{A \\ \text{quadratic growth} \\ \frac{1}{2}\begin{pmatrix} t-x & y \\ y & t+x \end{pmatrix}} \arrow[ur] \arrow[dr] & \\
& \substack{C_A \\ \text{circle} \\ \text{radius } t \text{ and center } (x,y)}
\end{tikzcd}
$$

lattices and circles

$$\Lambda_A = \Lambda_M$$

lattice

$\{\mathbf{v} \in \mathbb{Z}^2 : A\mathbf{v} \in \mathbb{Z}^2\}$

$\beta\mathbb{Z} + \delta\mathbb{Z}$

$$A$$

quadratic growth

$$\frac{1}{2}\begin{pmatrix} t-x & y \\ y & t+x \end{pmatrix}$$

$$M$$

Möbius transformation

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

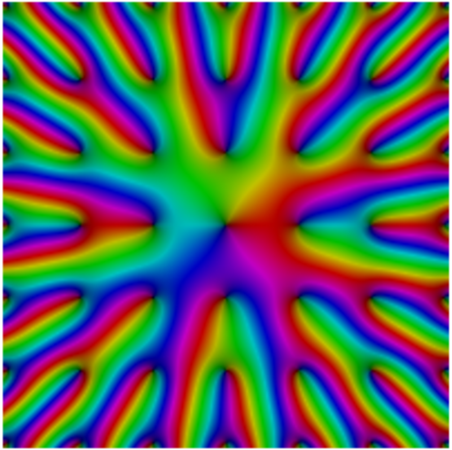
$$C_A = C_M$$

circle

radius t and center (x, y)

$M \cdot \widehat{\mathbb{R}}$

theta functions

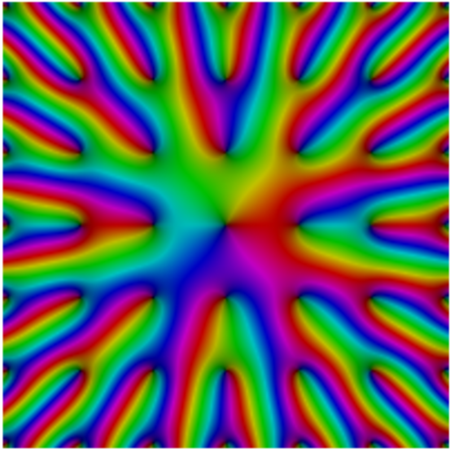


Given a lattice $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}$, one asks for the *elliptic functions* for that lattice, i.e. meromorphic and periodic.

The solution is given by *theta functions*, e.g. the *Weierstrass σ-function*, which has an automorphy factor, i.e. for $\lambda \in \Lambda_\tau$,

$$\sigma(z+\lambda;\tau) = \sigma(z;\tau)\psi(\lambda)e^{\langle \lambda, z\rangle_\tau + \frac{1}{2}\langle \lambda,\lambda\rangle_\tau}.$$

theta functions



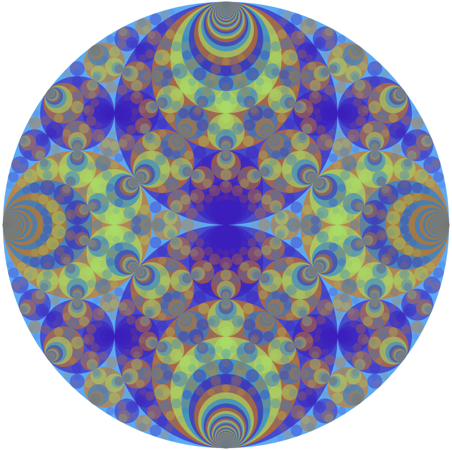
For each peak of Γ, there exists a theta-like choice of odometer:

$$g(x+\lambda) = g(x) + x^T A\lambda + g(\lambda), \lambda \in \Lambda_A.$$

Associated to $\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$, one has a certain theta function whose q-valuation (taking the q-expansion) gives rise to this same odometer g.

This is work in progress…

thank you for your time and attention



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