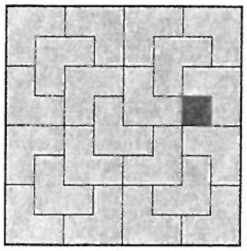
Note that $4^{k+1} - 1 = 4 \cdot 4^k - 1 = 4(4^k - 1) + 4 - 1 = 4(4^k - 1) + 3$. Since $4^k - 1$ and 3 are both divisible by 3, it follows that $4(4^k - 1) + 3$ is divisible by 3 hence $4^{k+1} - 1$ is divisible by 3. ∎

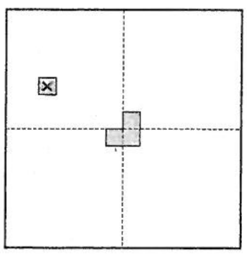


The next example involves some geometry. We wish to cover a chess board with special tiles called *L-shaped triominoes*, or *L-triominoes* for short. These are tiles formed from three 1×1 squares joined at their edges to form an L shape.

It is not possible to tile a standard 8×8 chess board with L-triominoes because there are 64 squares on the chess board and 64 is not divisible by 3. However, it is possible to cover all but one square of the chess board, and such a tiling is shown in the figure.

Is it possible to tile larger chess boards? A $2^n \times 2^n$ chess board has 4^n squares, so, applying Proposition 22.7, we know that $4^n - 1$ is divisible by 3. Hence there is a hope that we may be able to cover all but one of the squares.

Proposition 22.8 Let n be a positive integer. For every square on a $2^n \times 2^n$ chess board, there is a tiling by L-triominoes of the remaining $4^n - 1$ squares.



Proof. The proof is by induction on n. The basis case, $n = 1$, is obvious since placing an L-triomino on a 2×2 chess board covers all but one of the squares, and by rotating the triomino we can select which square is missed.

Suppose (induction hypothesis) that the Proposition has been proved for $n = k$.

We are given a $2^{k+1} \times 2^{k+1}$ chess board with one square selected. Divide the board into four $2^n \times 2^n$ subboards (as shown); the selected square must lie in one of these subboards. Place an L-triomino overlapping three corners from the remaining subboards as shown in the diagram.

We now have four $2^k \times 2^k$ subboards each with one square that does not need to be covered. By induction, the remaining squares in the subboards can be tiled by L-triominoes. ∎

Strong Induction

Here is a variation on Theorem 22.2.

Theorem 22.9 **(Principle of Mathematical Induction—strong version)** Let A be a set of natural numbers. If

- $0 \in A$ and
- for all $k \in \mathbb{N}$, if $0, 1, 2, \ldots, k \in A$, then $k + 1 \in A$

then $A = \mathbb{N}$.

The proof of this theorem is left to you (see Exercise 22.23).

Why is this called *strong* induction? Suppose you are using induction to prove a proposition. In both standard and strong induction, you begin by showing the basis case ($0 \in A$). In standard induction, you assume the induction hypothesis ($k \in A$; i.e., the proposition is true for $n = k$) and then use that to prove $k + 1 \in A$ (i.e., the proposition is true for $n = k + 1$). Strong induction gives you a stronger induction hypothesis. In strong induction, you may assume $0, 1, 2, \ldots, k \in A$ (the proposition is true for all n from 0 to k) and use that to prove $k + 1 \in A$ (the proposition is true for $n = k + 1$).

This method is outlined in Proof Template 18.