Making Change by Induction

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Theorem 1. *Every integer number $n \geq 4$ of cents can be formed from 2 and 5 cent coins.*

Proof by regular induction, standard base case. **Base case: 4 cents can be formed from 2 and 5 cent coins.**
Proof: Two 2-cent coins form a total of 4 cents.

Inductive Step: If $k \geq 4$ cents can be formed from 2 and 5 cent coins, then $k+1$ cents can be formed from 2 and 5 cent coins.
Proof: Our goal is to form $k+1$ cents using 2 and 5 cent coins. First, form k cents on the counter using 2 and 5 cent coins.

Claim: The coins used include either a 5 cent coin or two 2 cent coins. *Proof of Claim:* If not, then the coins include at most one 2 cent coin and no 5 cent coins. But then $k < 4$, which is a contradiction, proving the claim.

So we divide into two cases:

1. **Case I: The coins on the counter include a 5 cent coin.** In this case, remove the 5 cent coin and replace it with three 2-cent coins.

2. **Case II: The coins on the counter include two 2 cent coins.** In this case, remove the two 2 cent coins and replace them with a 5-cent coin.

The counter now has $k+1$ cents on it, formed using 2 and 5 cent coins. □

Proof by Strong Induction, adapted base case. **Base case: 4 or 5 cents can be formed from 2 and 5 cent coins.**
Proof: Two 2-cent coins form a total of 4 cents. One 5-cent coin forms a total of 5 cents.

Strong Inductive Step: If $r \geq 4$ cents can be formed from 2 and 5 cent coins for any $4 \leq r \leq k$, then $k+1$ cents can be formed from 2 and 5 cent coins.
Proof: Our goal is to form $k+1$ cents using 2 and 5 cent coins. We have $k+1 \geq 6$ since we are not in the base case, so $k-1 \geq 4$. This means we may use the inductive hypothesis on $r := k-1$. Form $k-1$ cents on the counter from 2 and 5 cent coins. Now add a 2-cent coin. The counter now has $k+1$ cents on it, formed using 2 and 5 cent coins. □