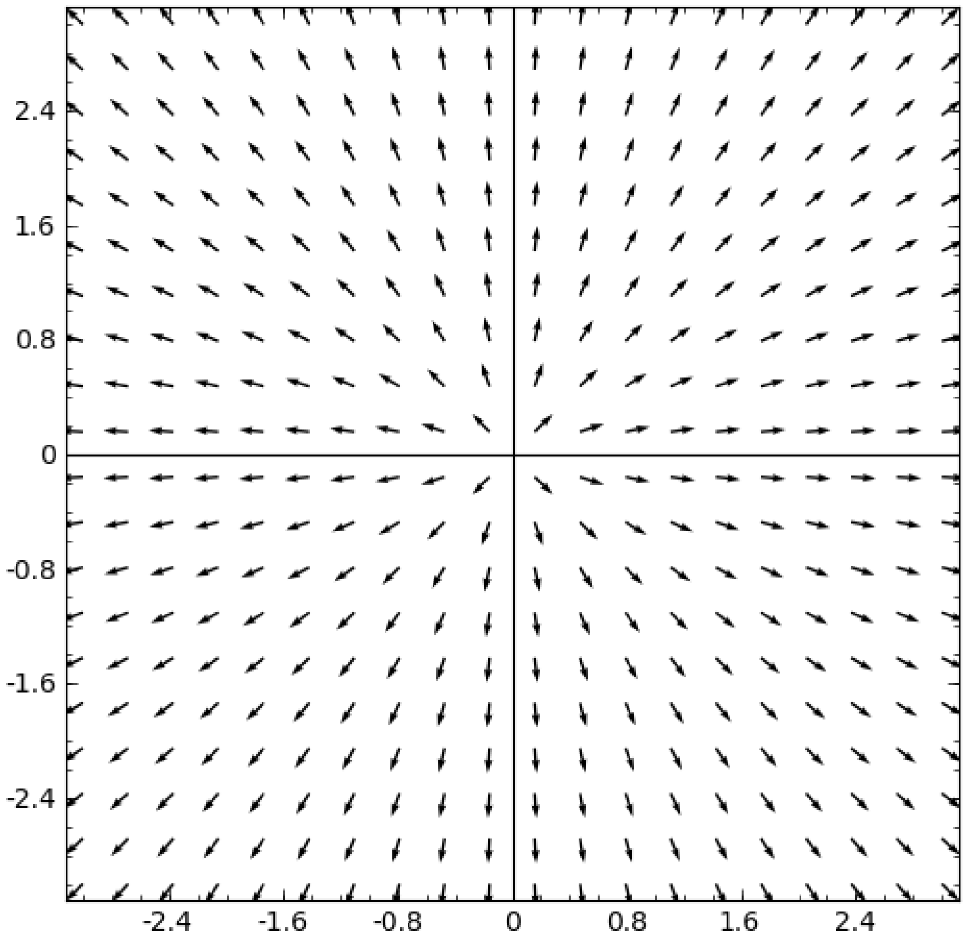
Homework 7 Extra Problems - Solutions

Problem 1



a) Since the vectors point out of the origin in the direction of their position, and are all the same length, this is the vector field

$$\mathbf{F}(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|} = \langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \rangle.$$

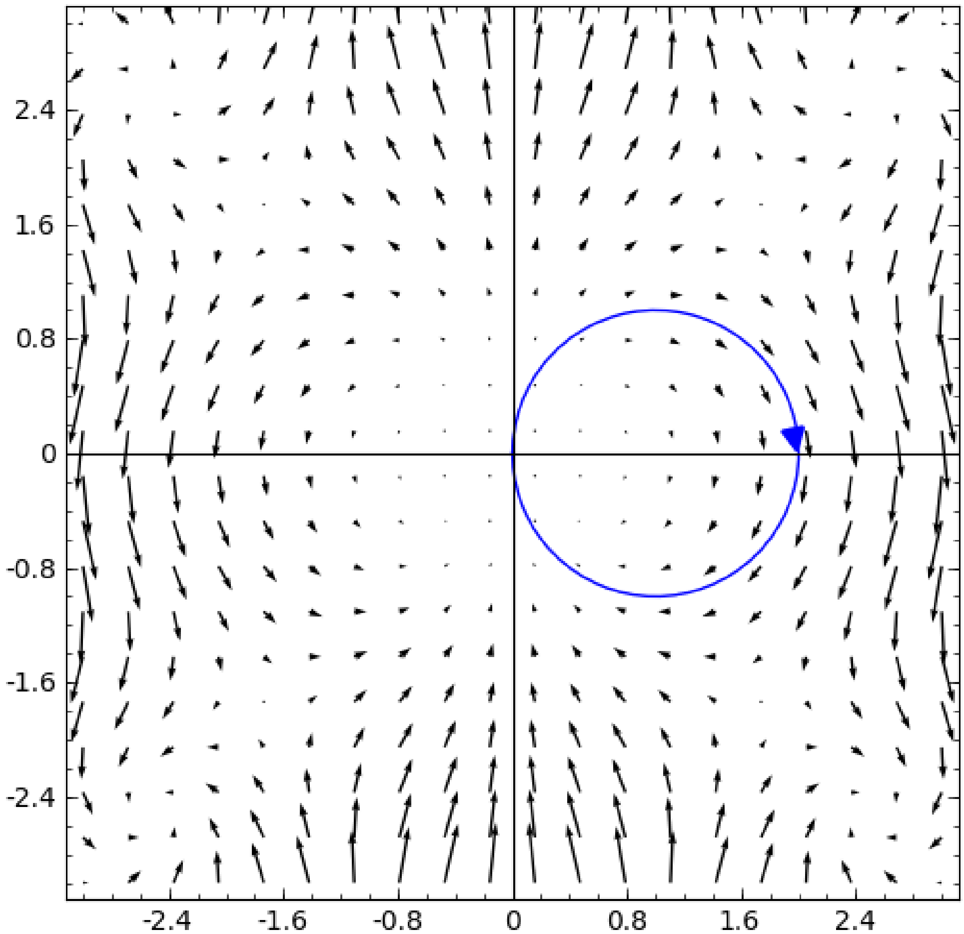
b) There are many possible correct answers. The simplest is the unit circle. Let $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$. Then $\mathbf{T}$ is perpendicular to $\mathbf{F}$ at each point, hence the dot product is always zero, and we have $\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C 0 ds = 0$.

c) There are many possible correct answers. To get a positive integral, it suffices to pick a C so that $\mathbf{F}$ always has a positive component in the direction of $\mathbf{T}$. A line segment along the x axis from the origin to $(1,0)$ will do. This segment has $\mathbf{r}(t) = \langle t, 0 \rangle$ for $0 \le t \le 1$.

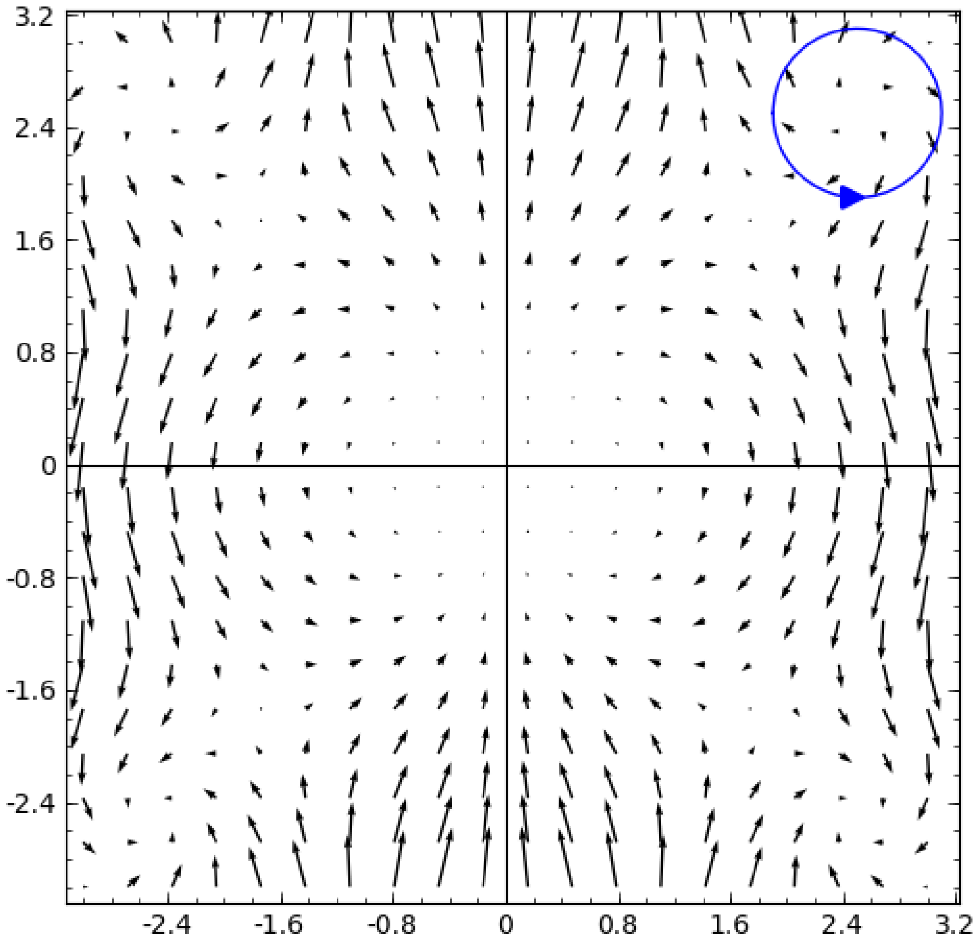
d) There are many possible correct answers. To get a negative integral, we can reverse the C for part c, i.e. take a line segment along the x axis from $(1,0)$ to the origin. Then $\mathbf{F}$ is in a direction opposite the direction of $\mathbf{T}$ all along this curve, and the integral will be negative. This segment has $\mathbf{r}(t) = \langle 1-t, 0 \rangle$ for $0 \le t \le 1$.

Problem 2

a) There are many possible correct answers. The following is one example. Any example where the vector field arrows tend to line up along the direction of the curve rather than against it will be correct.

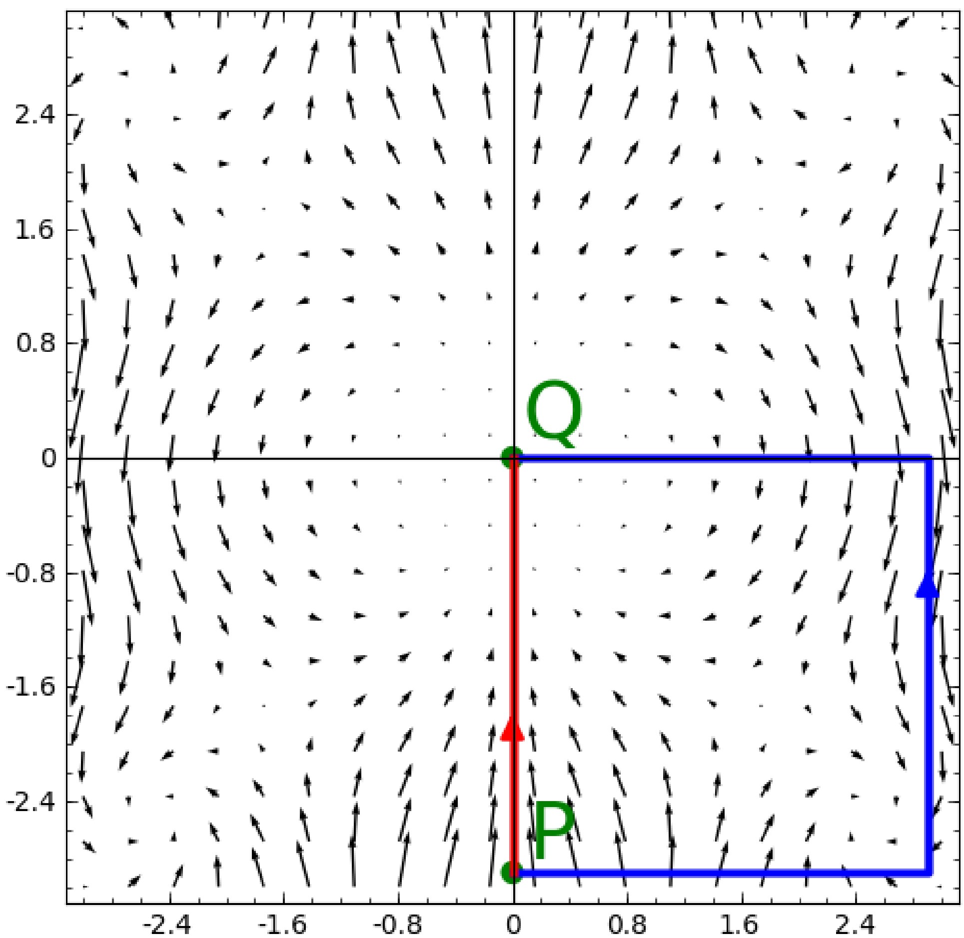


b) There are many correct answers. It would suffice to take the example above, only with opposite orientation. However, the question asks for a non-intersecting curve. So here's a more interesting one. It surrounds a sort of vortex, but the orientation is opposite the direction of the field arrows. Anything where the field arrows tend to line up against the direction of the curve will be correct.



c) There are many correct answers. Examples could be constructed from parts (a) or (b). But here's a more interesting one. In the image below, P and Q are labelled. The blue path is C_3 and the red (shorter, more direct) path is C_4. Along the red path, the direction of $\mathbf{F}$ is generally in the direction of $\mathbf{T}$. So the integral is positive. The blue path has three segments, and along each, respectively, we find that $\mathbf{F}$ is 1) generally perpendicular to the path; 2) generally opposite the direction of the path; and 3) generally perpendicular to the path. So we expect this integral to come out negative. Hence we obtain

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} < \int_{C_4} \mathbf{F} \cdot d\mathbf{r}$$



Problem 3

Let $\mathbf{F}$ be a constant vector field. Then it has the form $\mathbf{F}(x,y) = \mathbf{c} = \langle c_1, c_2 \rangle$. Let C be a closed loop parametrised by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ for $a \leq t \leq b$. Then

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathbf{r} &= \int_a^b \mathbf{F}(x(t), y(t)) \cdot \mathbf{r}'(t) dt && \text{by definition} \\
&= \int_a^b \langle c_1, c_2 \rangle \cdot \langle x'(t), y'(t) \rangle dt \\
&= \int_a^b c_1 x'(t) + c_2 y'(t) dt \\
&= \int_a^b c_1 x'(t) dt + \int_a^b c_2 y'(t) dt \\
&= c_1 x(t)|_a^b + c_2 y(t)|_a^b \\
&= c_1(x(b) - x(a)) + c_2(y(b) - y(a)) \\
&= c_1(0) + c_2(0) && \text{since } \mathbf{r}(a) = \mathbf{r}(b) \text{ since loop is closed} \\
&= 0
\end{aligned}$$

Therefore every line integral around a closed loop of the vector field is zero.