Homework 11 Extra Problem - Solutions

The average latitude is obtained by integrating the latitude with respect to area, then dividing by the area. (This is how we would average any quantity over a surface.)

The average latitude depends on the shape of a sphere but not on its radius, so we can assume the radius is 1 for simplicity. (One way to see this is to use units of "earth-radii" for the radius. Then the earth is 1 earth-radius in radius. Mars, by the way, is a little over 1/2 earth-radii in radius. Since the answer (average latitude) is an angle, i.e. has no units, it cannot depend on the units we use.)

Another note: I didn't say whether we use spherical coordinate latitude (measured down from north pole) or geographic latitude (measured up from equator), so I'll use the former. The average of both should be the same (as a physical latitude), and the answer can always be translated between the two if necessary.

The parametrisation of the northern hemisphere of the unit sphere is

$$\mathbf{r}(\phi,\theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle, \quad 0 \le \theta \le 2\pi, 0 \le \phi \le \pi/2$$

We have seen by a calculation in class that

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin\phi$$

So the integral of latitude (i.e. ϕ) is

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} \phi\sin\phi \, d\phi d\theta$$

To do the interior integral requires integration by parts.

$$\begin{aligned}\int_{\phi=0}^{\pi/2} \phi\sin\phi\, d\phi &= \int_{\phi=0}^{\pi/2} \cos\phi\, d\phi + \phi\cos\phi\Big|_0^{\pi/2} \\ &= -\sin\phi\Big|_0^{\pi/2} + \phi\cos\phi\Big|_0^{\pi/2} = 1\end{aligned}$$

The outer integral is trivial, so

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} \phi\sin\phi\, d\phi d\theta = 2\pi$$

We can also find the area of the upper hemisphere. It is

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} \sin\phi\, d\phi d\theta = 2\pi$$

(I'll leave it to you to check that calculation.) Therefore the average latitude is

$$\frac{\text{integral of latitude over northern hemisphere}}{\text{area of northern hemisphere}} = \frac{\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} \phi\sin\phi\, d\phi d\theta}{\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} \sin\phi\, d\phi d\theta} = \frac{2\pi}{2\pi} = 1$$

This seems very odd at first glance (where's the π gone?), but if you were asked to guess ahead of time what was the average latitude, you would probably say somewhere a bit below $\pi/4$ (since there's more surface area near the equator per degree of latitude than near the poles), maybe around $\pi/3$, which is not far from 1. Fascinating!