Problem 1 (5 parts, 30 points): Consider the curve

$$\mathbf{r}(t) = \left\langle 3\sin(t^2), 4t^2 + 7, 3\cos(t^2) \right\rangle, 0 \le t < \infty$$

a) (5 points) Find the arclength function $s(t)$ giving arclength along $\mathbf{r}(t)$ from the point at $t = 0$ in the direction of increasing t.

Solution. The arclength is the accumulation of the speed $|\mathbf{r}'(t)|$, i.e.

$$s(t) = \int_0^t |\mathbf{r}'(T)| dT.$$

(Here I've used variable T in the integral to prevent the confusion of writing $\int_0^t |\mathbf{r}'(t)| dt$. If we write this, we are using t two different ways. I generally forgave this on your exam. Your textbook uses u but we have a similar variable, $\mathbf{u}$, in the next part of Problem 1, so I chose T. You may use whatever you like.)

To compute this, we first find $|\mathbf{r}'(t)|$:

$$\mathbf{r}'(t) = \left\langle 6t\cos(t^2), 8t, -6t\sin(t^2) \right\rangle.$$

$$\begin{aligned} |\mathbf{r}'(t)| &= \sqrt{36t^2\cos^2(t^2) + 64t^2 + 36t^2\sin^2(t^2)} \\ &= \sqrt{100t^2} && \text{since } \cos^2(t^2) + \sin^2(t^2) = 1 \\ &= 10t \end{aligned}$$

Note: When I take this square root, I choose the positive root because the length of any vector must be non-negative. (In other words, the length formula always uses the positive square root.)

Then

$$\begin{aligned} s(t) &= \int_0^t |\mathbf{r}'(T)| dT \\ &= \int_0^t 10T dT \\ &= 5T^2|_0^t \\ &= 5t^2 \end{aligned}$$

One of the most common errors in this problem was forgetting the limits of integration in the formula for arclength. The observation which determines the limits is the following: the quantity $s(t)$ is the length along the curve from 0 to t.

$$\boxed{s(t) = 5t^2}$$

b) (5 points) Reparameterize $\mathbf{r}(t)$ with respect to the arclength from the point at $t = 0$ in the direction of increasing t (i.e. the arclength of part (a)). Write your answer as $\mathbf{u}(s)$ and indicate the domain of s.

Solution. Taking $s = 5t^2$ from part (a), we solve for t:

$$\begin{aligned} s &= 5t^2 \\ s/5 &= t^2 \\ \sqrt{s/5} &= t \end{aligned}$$

Plugging in $t(s) = \sqrt{s/5}$ to $\mathbf{r}(t)$, we obtain

$$\mathbf{u}(s) = \mathbf{r}(t(s)) = \langle 3\sin\left(\frac{s}{5}\right), \frac{4}{5}s + 7, 3\cos\left(\frac{s}{5}\right)\rangle$$

To determine the domain: The functions $\mathbf{u}(s)$ and $\mathbf{r}(t)$ trace out the same curve in space. The domain for $\mathbf{r}(t)$ is $0 \leq t < \infty$ (meaning the curve starts at $t = 0$). When $t = 0$, $s = s(t) = s(0) = 0$. As t increases through positive numbers, $s(t) = 5t^2$ increases through positive numbers also. So the domain $0 \leq t < \infty$ “translates into” a domain of $0 \leq s < \infty$. In other words, our new parameter s goes from 0 upwards as our old parameter t goes from 0 upwards, tracing out the same curve.

Some students said s must be non-negative because $s = 5t^2$ is always non-negative. This is true, but I would also like you to think about the solution method outlined in the last paragraph.

I did not accept any answer that said $s = \infty$ was part of the domain. The symbol ∞ does not denote a number.

Bonus problem (no credit, but good to think about): What would it mean if I have given the domain of t as $-\infty < t < \infty$? Could you calculate arclength from the point $t = -1$, and could you reparametrize with respect to arclength? If so, what would the domain of s be? Why is this fishy?

$$\boxed{\mathbf{u}(s) = \langle 3\sin\left(\tfrac{s}{5}\right), \tfrac{4}{5}s + 7, 3\cos\left(\tfrac{s}{5}\right)\rangle}$$

$$\boxed{\text{domain of } s\text{: } 0 \leq s < \infty}$$

c) (8 points, no justification necessary) Since $\mathbf{r}(t)$ and $\mathbf{u}(s)$ parametrize the same space curve, we can consider a point P on that curve. What follows is a list of quantities which can be defined at the point P on $\mathbf{r}(t)$ or on $\mathbf{u}(s)$. Circle those items which are *necessarily the same* for both $\mathbf{r}(t)$ and $\mathbf{u}(s)$ at P.

$\boxed{\text{curvature}}$, tangent vector, $\boxed{\text{unit tangent vector}}$, $\boxed{\text{osculating circle}}$,

$\boxed{\text{binormal vector}}$, velocity, tangential acceleration, normal acceleration

d) (10 points) Compute the curvature κ of either $\mathbf{r}(t)$ or $\mathbf{u}(s)$ at the point $(0, 4\pi + 7, -3)$.

Solution. Many students chose the most difficult way to do this problem. More on that after two efficient solutions are shown here.

Solution #1 We will use the formula

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

because we have already computed the denominator in part (a) of this Problem (it was $10t$). To compute the numerator, begin with $\mathbf{r}(t)$ computed from the first problem, and find its unit vector:

$$\mathbf{T}(t) = \frac{\mathbf{r}(t)}{|\mathbf{r}(t)|} = \frac{1}{10t}\langle 6t\cos(t^2), 8t, -6t\sin(t^2)\rangle = \left\langle \frac{3}{5}\cos(t^2), \frac{4}{5}, -\frac{3}{5}\sin(t^2)\right\rangle.$$

The derivative of this vector isn't so bad, it is

$$\mathbf{T}'(t) = \left\langle -\frac{6}{5}t\sin(t^2), 0, -\frac{6}{5}t\cos(t^2)\right\rangle.$$

Since $\cos^2(t^2) + \sin^2(t^2) = 1$, we find

$$|\mathbf{T}'(t)| = \frac{6}{5}t$$

Then

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\frac{6}{5}t}{10t} = \frac{3}{25}.$$

Solution #2 The method of the last solution can be made even shorter if you use $\mathbf{u}(s)$ instead. There we know that $|\mathbf{u}'(s)| = 1$ because that's the purpose of the arclength reparametrisation: it makes the speed of the 'particle' exactly 1 (for those who observed this instead of calculating it, I gave a bonus point). The function $\mathbf{u}(s)$ is also easier to take derivatives of than $\mathbf{r}(t)$ (fractions aren't as bad as they look – constants behave well under differentiation – but products like $t\sin(t)$ mean costly product rule computations). Furthermore, since $\mathbf{u}'(s)$ is a unit vector, $\mathbf{T}'(s) = \mathbf{u}''(s)$ for this function. We compute

$$\mathbf{u}'(s) = \langle \frac{3}{5}\cos(s/5), 4/5, -\frac{3}{5}\sin(s/5)\rangle$$
$$\mathbf{u}''(s) = \langle -\frac{3}{25}\sin(s/5), 0, -\frac{3}{25}\cos(s/5)\rangle$$

Finally,

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = |\mathbf{u}''(s)| = \frac{3}{25}$$

Important Remark. In both these cases, we never used the point! This curve is a helix (as you can see from the formula for $\mathbf{u}(s)$), up to some constants. And helices have constant curvature. So it's perfectly alright that we never used the point. This observation should help you with one of the True/False questions.

For those of you who used the formula

$$\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

the day could still be saved if you plugged in the value for t at the given point ($t = \sqrt{\pi}$) before you did all the cross product, etc. But the algebra in this case is still a big pain, and most of you who used this method did not succeed in getting it down to the simplified final answer without making errors.

For future reference. I do not like to give difficult algebra on an exam, since it just results in wasted time and mistakes that derail you before you can show that you understand the theory. If you find yourself doing difficult algebra, it's because you are doing the question *the hard way*. Stop, take a breath, think about it, and pick a better approach. (Sadly, it seems that almost any question has a hard way to tempt you down the wrong path. Beware!)

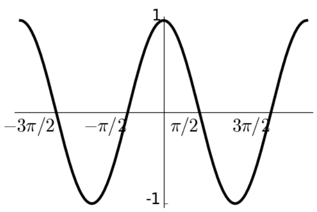
$$\boxed{\kappa = \frac{3}{25}}$$

e) (2 points, no partial credit, no justification needed) What is the radius of the osculating circle at the point $(0, 4\pi + 7, -3)$? Circle the correct answer:

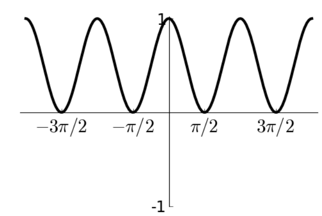
Remark. In this problem, I gave you full points if you put $1/\kappa$ where κ was whatever curvature you got in the last part (since the radius is the inverse of the curvature).

1/60, 1/25, 3/25, 4/25, 1/5, 1/2, 1, 2, 5, 25/4, $\boxed{25/3}$, 25, 60, none of these.

Problem 2 (24 points, no justification needed): Consider the following plane curves (a) through (o) and the space curves (A) through (F).



a)



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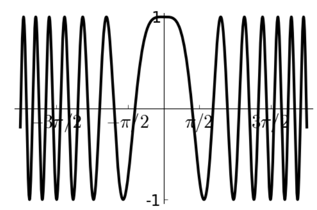
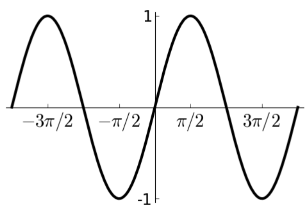
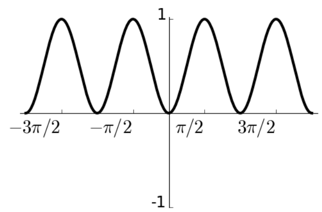


c)



d)



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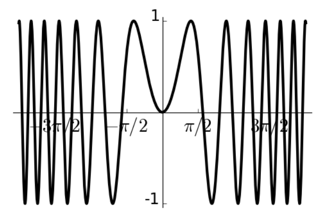
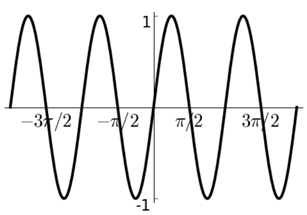
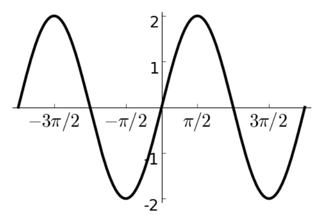


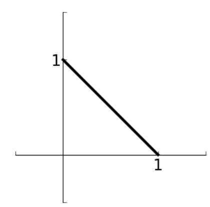
f)



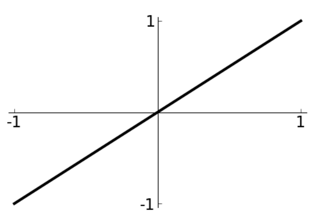
g)



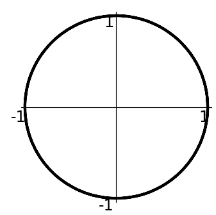
h)



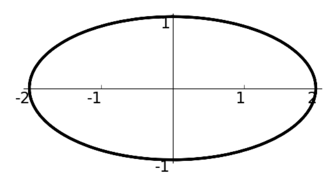
i)



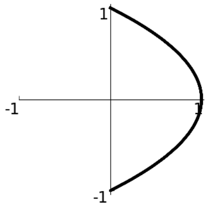
j)



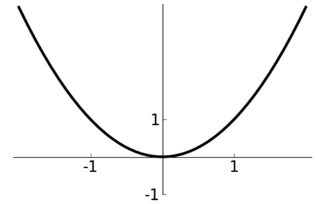
k)



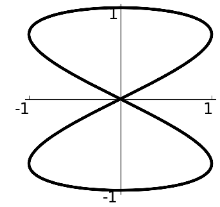
l)



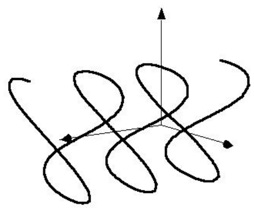
m)



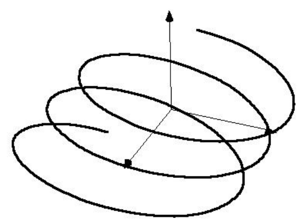
n)



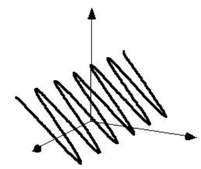
o)



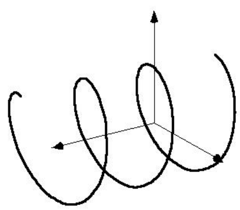
A)



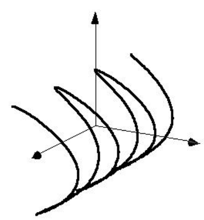
B)



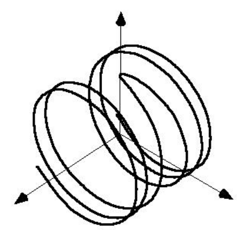
C)



D)



E)



F)

In the table below, fill in the letter corresponding to the correct plane or space curve in each of the empty boxes. It is possible that you may need to use some graphs repeatedly, and some not at all.

Remark. Different exams had different functions on them. I've expanded the table here to include all those functions and more. It makes a good study tool: cover up the answers with another sheet of paper, and then do the problem for all six functions shown below.

	viewed down x axis	viewed down y axis	viewed down z axis	graph of space curve
$\mathbf{r}(t) = \langle t, \sin(t), \cos(t)\rangle$	k	a	d	D
$\mathbf{r}(t) = \langle t, \sin^2(t), \cos^2(t)\rangle$	i	b	e	C
$\mathbf{r}(t) = \langle t, \sin(2t), \cos(t)\rangle$	o	a	g	A
$\mathbf{r}(t) = \langle t, \sin^2(t), \cos(t)\rangle$	m	a	e	E
$\mathbf{r}(t) = \langle t, 2\sin(t), \cos(t)\rangle$	l	a	h	B
$\mathbf{r}(t) = \langle t, \sin(t^2), \cos(t^2)\rangle$	k	c	f	F

Problem 3 (14 points, no justification needed): Say whether the following statements are true (**T**) or false (**F**). You may assume that all functions are defined everywhere and have derivatives of all orders everywhere. You do not need to give reasons; this problem will be graded by answer only. **You will get +2 for each correct answer, 0 for each wrong answer, and +1 for each unanswered.**

T F If a particle with position function $\mathbf{r}(t)$ experiences no acceleration, then its arclength function $s(t)$ is linear in t.

True If the particle experiences no acceleration, then it is moving with constant velocity. Arclength $s(t)$ is the distance travelled along the curve in time t. If it has constant speed v, that distance is increasing at a constant rate, i.e. $s(t)$ is linear, or in other words $s(t) = vt + c$ for some constant c depending on where you are measuring from. Here's another way to think of it: If $\mathbf{r}(t)$ has zero acceleration, then $\mathbf{r}''(t) = 0$. Antidifferentiating, we find that $\mathbf{r}'(t)$ must be a constant. Hence $|\mathbf{r}'(t)|$ is equal to some constant k. Calculating $s(t)$ as an integral of this function (by the arclength formula), we find $s(t) = kt + c$ for some constant c (depending on where we are measuring arclength from).

T F If a space curve lies entirely in a plane, then the binormal vector is always normal to that plane.

True If a curve lies in a plane, then the unit tangent $\mathbf{T}$ lies in that plane. The normal vector $\mathbf{N}$ is also in that plane: since $\mathbf{N}$ measures the change in $\mathbf{T}$, it lies in the plane $\mathbf{T}$ lies in. Since $\mathbf{B}$ is perpendicular to $\mathbf{T}$ and $\mathbf{N}$, it is perpendicular to the plane they generate.

T F If $\mathbf{r}(t)$ is a space curve (a curve in three dimensional space) with constant curvature κ, then $\mathbf{r}(t)$ parameterizes part of a circle of radius $1/\kappa$.

False. We saw in class and earlier in this exam that the helix is a curve with constant curvature. But the helix is not a part of a circle.

T F If a particle moves at constant speed, then the velocity and acceleration vectors are orthogonal.

True. The tangential component of acceleration is v' (derivative of speed), which is zero for a particle of constant speed. So acceleration is entirely in the normal component, i.e. in the direction of $\mathbf{N}$ which is perpendicular to $\mathbf{v}$.

T F To determine the velocity of a particle, it suffices to know its acceleration.

False. To obtain velocity from acceleration requires an antiderivative, which results in a constant '$+c$'. For example, $\mathbf{r}(t) = \langle t^2 + t, 1\rangle$ and $\mathbf{r}(t) = \langle t^2 + 2t, 1\rangle$ have different velocities, but the same acceleration.

T F If the tangential component of acceleration for a particle is constantly 4, then that particle travels on a path of constant curvature.

False. This is just nonsense. Knowing the tangential component (v') is constant tells us something about the speed the particle moves, but not about the direction it moves (including the curviness of its path). For example, take any curve that is not of constant curvature (e.g. $\mathbf{r}(t) = \langle t, t^2\rangle$), and reparametrise it to make the particle have speed $4t$ (actually working this out is difficult, but in theory it is possible to reparametrise any curve to have any speed).

T F The osculating circle of a curve C at a point has the same unit tangent vector, normal vector, and curvature as C at that point.

True. This is the definition.

Problem 4 (10 points): Show that for a space curve, $\frac{d\mathbf{B}}{ds}$ is perpendicular to $\mathbf{T}$.

Proof #1

To check that $\frac{d\mathbf{B}}{ds}$ is perpendicular to $\mathbf{T}$, it suffices to check that their dot product is zero. In other words, we will have proven the claim if we show that $\frac{d\mathbf{B}}{ds} \cdot \mathbf{T} = 0$.

Since $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ be definition, we have

$$\begin{aligned} \frac{d\mathbf{B}}{ds} &= \frac{d(\mathbf{T} \times \mathbf{N})}{ds} \\ &= \frac{d\mathbf{T}}{ds} \times \mathbf{N} + \mathbf{T} \times \frac{\mathbf{N}}{ds} \qquad \text{by cross product rule for differentiation} \end{aligned}$$

Taking the dot product with $\mathbf{T}$ on both sides, we have

$$\begin{aligned} \frac{d\mathbf{B}}{ds} \cdot \mathbf{T} &= \left(\frac{d\mathbf{T}}{ds} \times \mathbf{N} + \mathbf{T} \times \frac{\mathbf{N}}{ds}\right) \cdot \mathbf{T} \\ &= \left(\frac{d\mathbf{T}}{ds} \times \mathbf{N}\right) \cdot \mathbf{T} + \left(\mathbf{T} \times \frac{\mathbf{N}}{ds}\right) \cdot \mathbf{T} \qquad \text{since dot product distributes over sum} \end{aligned}$$

We will show that both these terms vanish (are zero).

First, we show the term $\left(\mathbf{T} \times \frac{d\mathbf{N}}{ds}\right) \cdot \mathbf{T}$ vanishes, as follows. A cross product $\mathbf{a} \times \mathbf{b}$ is perpendicular to the two input vectors $\mathbf{a}$ and $\mathbf{b}$. Here, $\mathbf{T} \times \frac{d\mathbf{N}}{ds}$ is perpendicular to $\mathbf{T}$. Therefore their dot product vanishes.

Second, we show the term $\left(\frac{d\mathbf{T}}{ds} \times \mathbf{N}\right) \cdot \mathbf{T}$ vanishes, as follows. Recall that the curvature vector is $\frac{d\mathbf{T}}{ds}$ and that this vector is equal to $\kappa\mathbf{N}$. Hence the term becomes $(\kappa\mathbf{N} \times \mathbf{N}) \cdot \mathbf{T}$. A cross product of any vector with another parallel vector is zero. But $\mathbf{N}$ and $\kappa\mathbf{N}$ are parallel. So $\kappa\mathbf{N} \times \mathbf{N}$ is zero. Finally, the dot product of the zero vector with any vector is zero. So the term vanishes.

Since both terms vanish, we have shown

$$\begin{aligned} \frac{d\mathbf{B}}{ds} \cdot \mathbf{T} &= \left(\frac{d\mathbf{T}}{ds} \times \mathbf{N}\right) \cdot \mathbf{T} + \left(\mathbf{T} \times \frac{\mathbf{N}}{ds}\right) \cdot \mathbf{T} \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Remark on the proof. This is the proof that easiest to find, in the sense that the method of proof is direct: try to compute $\frac{d\mathbf{T}}{ds} \cdot \mathbf{T}$ and see if you get 0. The next proof is harder to find, but shorter.

Proof #2 (bonus for this one)

A cross product is perpendicular to the two input vectors. Since $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, we find that $\mathbf{B}$ is perpendicular to $\mathbf{T}$ and $\mathbf{N}$. This has two consequences. The first is that since perpendicular vectors have zero dot product,

$$0 = \mathbf{B} \cdot \mathbf{T} \tag{1}$$

The next consequence is as follows. The term $\frac{d\mathbf{T}}{ds}$ is the curvature vector, which is equal to $\kappa\mathbf{N}$. Since $\mathbf{N}$ and $\mathbf{B}$ are perpendicular (from the first paragraph), it follows that $\frac{d\mathbf{T}}{ds}$ and $\mathbf{B}$ are perpendicular. Thus

$$\mathbf{B} \cdot \frac{d\mathbf{T}}{ds} = 0 \tag{2}$$

Differentiating (1), we find that

$$\begin{aligned} 0 &= \frac{d(\mathbf{B} \cdot \mathbf{T})}{ds} \\ &= \frac{d\mathbf{B}}{ds} \cdot \mathbf{T} + \mathbf{B} \cdot \frac{d\mathbf{T}}{ds} && \text{by dot product rule of differentiation} \\ &= \frac{d\mathbf{B}}{ds} \cdot \mathbf{T} && \text{by (2)} \end{aligned}$$

From the vanishing of the dot product, we find that $\frac{d\mathbf{B}}{ds}$ is perpendicular to $\mathbf{T}$.