**20.** $\mathbf{r}_u = \cos v\,\mathbf{i} + \sin v\,\mathbf{j}$, $\mathbf{r}_v = -u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j} + \mathbf{k} \quad\Rightarrow\quad \mathbf{r}_u \times \mathbf{r}_v = \sin v\,\mathbf{i} - \cos v\,\mathbf{j} + u\,\mathbf{k}$ and $\mathbf{F}(\mathbf{r}(u,v)) = u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j} + v^2\,\mathbf{k}$. Then by Formula 9,

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA = \int_0^\pi\int_0^1 (u\sin^2 v - u\cos^2 v + uv^2)\,du\,dv\\ &= \int_0^\pi\int_0^1(-u\cos 2v + uv^2)\,du\,dv = \int_0^\pi \left[-\tfrac{1}{2}\cos 2v + \tfrac{1}{2}v^2\right]dv = \tfrac{1}{6}\pi^3\end{aligned}$$

**22.** $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z^4\,\mathbf{k}$, $z = g(x,y) = \sqrt{x^2+y^2}$, and $D$ is the disk $\{(x,y) \mid x^2+y^2 \le 1\}$. Since $S$ has downward orientation, we have

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= -\iint_D \left[-x\left(\frac{x}{\sqrt{x^2+y^2}}\right) - y\left(\frac{y}{\sqrt{x^2+y^2}}\right) + z^4\right] dA = -\iint_D \left[\frac{-x^2-y^2}{\sqrt{x^2+y^2}} + \left(\sqrt{x^2+y^2}\right)^4\right] dA\\ &= -\int_0^{2\pi}\int_0^1 \left(\frac{-r^2}{r} + r^4\right) r\,dr\,d\theta = -\int_0^{2\pi} d\theta \int_0^1 (r^5 - r^2)\,dr = -2\pi\left(\tfrac{1}{6} - \tfrac{1}{3}\right) = \tfrac{\pi}{3}\end{aligned}$$

**28.** Here $S$ consists of three surfaces: $S_1$, the lateral surface of the cylinder; $S_2$, the front formed by the plane $x + y = 2$; and the back, $S_3$, in the plane $y = 0$.

On $S_1$: $\mathbf{F}(\mathbf{r}(\theta,y)) = \sin\theta\,\mathbf{i} + y\,\mathbf{j} + 5\,\mathbf{k}$ and $\mathbf{r}_\theta \times \mathbf{r}_y = \sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{k} \quad\Rightarrow$

$$\begin{aligned}\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^{2-\sin\theta}(\sin^2\theta + 5\cos\theta)\,dy\,d\theta\\ &= \int_0^{2\pi}(2\sin^2\theta + 10\cos\theta - \sin^3\theta - 5\sin\theta\,\cos\theta)\,d\theta = 2\pi\end{aligned}$$

On $S_2$: $\mathbf{F}(\mathbf{r}(x,z)) = x\,\mathbf{i} + (2-x)\,\mathbf{j} + 5\,\mathbf{k}$ and $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{i} + \mathbf{j}$.

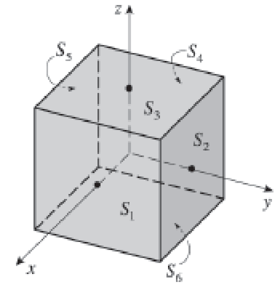
$$\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+z^2\le 1} [x + (2-x)]\,dA = 2\pi$$

On $S_3$: $\mathbf{F}(\mathbf{r}(x,z)) = x\,\mathbf{i} + 5\,\mathbf{k}$ and $\mathbf{r}_x \times \mathbf{r}_z = -\mathbf{j}$ so $\iint_{S_3} \mathbf{F}\cdot d\mathbf{S} = 0$. Hence $\iint_S \mathbf{F}\cdot d\mathbf{S} = 4\pi$.

**42.** A parametric representation for the hemisphere $S$ is $\mathbf{r}(\phi,\theta) = 3\sin\phi\,\cos\theta\,\mathbf{i} + 3\sin\phi\,\sin\theta\,\mathbf{j} + 3\cos\phi\,\mathbf{k}$, $0 \le \phi \le \pi/2$, $0 \le \theta \le 2\pi$. Then $\mathbf{r}_\phi = 3\cos\phi\,\cos\theta\,\mathbf{i} + 3\cos\phi\,\sin\theta\,\mathbf{j} - 3\sin\phi\,\mathbf{k}$, $\mathbf{r}_\theta = -3\sin\phi\sin\theta\,\mathbf{i} + 3\sin\phi\cos\theta\,\mathbf{j}$, and the outward orientation is given by $\mathbf{r}_\phi \times \mathbf{r}_\theta = 9\sin^2\phi\,\cos\theta\,\mathbf{i} + 9\sin^2\phi\,\sin\theta\,\mathbf{j} + 9\sin\phi\,\cos\phi\,\mathbf{k}$. The rate of flow through $S$ is

$$\begin{aligned}\iint_S \rho\mathbf{v}\cdot d\mathbf{S} &= \rho\int_0^{\pi/2}\int_0^{2\pi}(3\sin\phi\,\sin\theta\,\mathbf{i} + 3\sin\phi\,\cos\theta\,\mathbf{j})\cdot\left(9\sin^2\phi\,\cos\theta\,\mathbf{i} + 9\sin^2\phi\sin\theta\,\mathbf{j} + 9\sin\phi\,\cos\phi\,\mathbf{k}\right)d\theta\,d\phi\\ &= 27\rho\int_0^{\pi/2}\int_0^{2\pi}\left(\sin^3\phi\,\sin\theta\,\cos\theta + \sin^3\phi\,\sin\theta\,\cos\theta\right)d\theta\,d\phi = 54\rho\int_0^{\pi/2}\sin^3\phi\,d\phi\int_0^{2\pi}\sin\theta\,\cos\theta\,d\theta\\ &= 54\rho\left[-\tfrac{1}{3}(2+\sin^2\phi)\cos\phi\right]_0^{\pi/2}\left[\tfrac{1}{2}\sin^2\theta\right]_0^{2\pi} = 0\text{ kg/s}\end{aligned}$$

**44.** Referring to the figure, on




$S_1$: $\mathbf{E} = \mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i}$ and $\iint_{S_1} \mathbf{E} \cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} dy\,dz = 4$;

$S_2$: $\mathbf{E} = x\,\mathbf{i} + \mathbf{j} + z\,\mathbf{k}$, $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{j}$ and $\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} dx\,dz = 4$;

$S_3$: $\mathbf{E} = x\,\mathbf{i} + y\,\mathbf{j} + \mathbf{k}$, $\mathbf{r}_x \times \mathbf{r}_y = \mathbf{k}$ and $\iint_{S_3} \mathbf{E} \cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1} dx\,dy = 4$;

$S_4$: $\mathbf{E} = -\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, $\mathbf{r}_z \times \mathbf{r}_y = -\mathbf{i}$ and $\iint_{S_4} \mathbf{E} \cdot d\mathbf{S} = 4$.

Similarly $\iint_{S_5} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_6} \mathbf{E} \cdot d\mathbf{S} = 4$. Hence $q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S} = \varepsilon_0 \sum_{i=1}^{6} \iint_{S_i} \mathbf{E} \cdot d\mathbf{S} = 24\varepsilon_0$.

**2.** The boundary curve $C$ is the circle $x^2 + y^2 = 9$, $z = 0$ oriented in the counterclockwise direction when viewed from above. A vector equation of $C$ is $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$, so $\mathbf{r}'(t) = -3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j}$ and $\mathbf{F}(\mathbf{r}(t)) = 2(3\sin t)(\cos 0)\,\mathbf{i} + e^{3\cos t}(\sin 0)\,\mathbf{j} + (3\cos t)e^{3\sin t}\,\mathbf{k} = 6\sin t\,\mathbf{i} + (3\cos t)e^{3\sin t}\,\mathbf{k}$. Then, by Stokes' Theorem,

$$\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^{2\pi} (-18\sin^2 t + 0 + 0)\,dt = -18\left[\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right]_0^{2\pi} = -18\pi.$$

**6.** The boundary curve $C$ is the unit circle in the $yz$-plane. By Equation 3, $\iint_{S_1} \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \operatorname{curl}\mathbf{F} \cdot d\mathbf{S}$ where $S_1$ is the original hemisphere and $S_2$ is the disk $y^2 + z^2 \le 1$, $x = 0$.

$\operatorname{curl}\mathbf{F} = (x - x^2)\,\mathbf{i} - (y + e^{xy}\sin z)\,\mathbf{j} + (2xz - xe^{xy}\cos z)\,\mathbf{k}$, and for $S_2$ we choose $\mathbf{n} = \mathbf{i}$ so that $C$ has the same orientation for both surfaces. Then $\operatorname{curl}\mathbf{F} \cdot \mathbf{n} = x - x^2$ on $S_2$, where $x = 0$. Thus

$$\iint_{S_2} \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \iint\limits_{y^2+z^2 \le 1} (x - x^2)\,dA = \iint\limits_{y^2+z^2 \le 1} 0\,dA = 0.$$

Alternatively, we can evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$: $C$ with positive orientation is given by $\mathbf{r}(t) = \langle 0, \cos t, \sin t\rangle$, $0 \le t \le 2\pi$, and

$$\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left\langle e^{0(\cos t)}\cos(\sin t), (0)^2(\sin t), (0)(\cos t)\right\rangle \cdot \langle 0, -\sin t, \cos t\rangle\,dt = \int_0^{2\pi} 0\,dt = 0.$$

**8.** $\operatorname{curl}\mathbf{F} = e^x\,\mathbf{k}$ and $S$ is the portion of the plane $2x + y + 2z = 2$ over $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 2 - 2x\}$. We orient $S$ upward and use Equation 17.7.10 [ET 16.7.10] with $z = g(x, y) = 1 - x - \frac{1}{2}y$:

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} &= \iint_D (0 + 0 + e^x)\,dA = \int_0^1\int_0^{2-2x} e^x\,dy\,dx = \int_0^1 (2 - 2x)e^x\,dx \\ &= \left[(2 - 2x)e^x + 2e^x\right]_0^1 \quad \text{[integrating by parts]} \quad = 2e - 4\end{aligned}$$

**16.** Let $S$ be the surface in the plane $x+y+z=1$ with upward orientation enclosed by $C$. Then an upward unit normal vector for $S$ is $\mathbf{n}=\frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$. Orient $C$ in the counterclockwise direction, as viewed from above. $\int_C z\,dx-2x\,dy+3y\,dz$ is equivalent to $\int_C \mathbf{F}\cdot d\mathbf{r}$ for $\mathbf{F}(x,y,z)=z\,\mathbf{i}-2x\,\mathbf{j}+3y\,\mathbf{k}$, and the components of $\mathbf{F}$ are polynomials, which have continuous partial derivatives throughout $\mathbb{R}^3$. We have curl $\mathbf{F}=3\,\mathbf{i}+\mathbf{j}-2\,\mathbf{k}$, so by Stokes' Theorem,

$$\begin{aligned}\int_C z\,dx-2x\,dy+3y\,dz=\int_C \mathbf{F}\cdot d\mathbf{r}=\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS&=\iint_S(3\,\mathbf{i}+\mathbf{j}-2\,\mathbf{k})\cdot\frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})\,dS\\&=\frac{2}{\sqrt{3}}\iint_S dS=\frac{2}{\sqrt{3}}\,(\text{surface area of } S)\end{aligned}$$

Thus the value of $\int_C z\,dx-2x\,dy+3y\,dz$ is always $\frac{2}{\sqrt{3}}$ times the area of the region enclosed by $C$, regardless of its shape or location. [Notice that because $\mathbf{n}$ is normal to a plane, it is constant. But curl $\mathbf{F}$ is also constant, so the dot product curl $\mathbf{F}\cdot\mathbf{n}$ is constant and we could have simply argued that $\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS$ is a constant multple of $\iint_S dS$, the surface area of $S$.]

**20.** (a) By Exercise 17.5.26 [ET 16.5.26], $\operatorname{curl}(f\nabla g)=f\operatorname{curl}(\nabla g)+\nabla f\times\nabla g=\nabla f\times\nabla g$ since $\operatorname{curl}(\nabla g)=\mathbf{0}$. Hence by Stokes' Theorem $\int_C(f\nabla g)\cdot d\mathbf{r}=\iint_S(\nabla f\times\nabla g)\cdot d\mathbf{S}$.

(b) As in (a), $\operatorname{curl}(f\nabla f)=\nabla f\times\nabla f=\mathbf{0}$, so by Stokes' Theorem, $\int_C(f\nabla f)\cdot d\mathbf{r}=\iint_S[\operatorname{curl}(f\nabla f)]\cdot d\mathbf{S}=0$.

(c) As in part (a),

$$\begin{aligned}\operatorname{curl}(f\nabla g+g\nabla f)&=\operatorname{curl}(f\nabla g)+\operatorname{curl}(g\nabla f) &&[\text{by Exercise 17.5.24 [ET 16.5.24]}]\\&=(\nabla f\times\nabla g)+(\nabla g\times\nabla f)=\mathbf{0} &&[\text{since } \mathbf{u}\times\mathbf{v}=-(\mathbf{v}\times\mathbf{u})]\end{aligned}$$

Hence by Stokes' Theorem, $\int_C(f\nabla g+g\nabla f)\cdot d\mathbf{r}=\iint_S\operatorname{curl}(f\nabla g+g\nabla f)\cdot d\mathbf{S}=0$.