# Midterm #2 SOLUTIONS

Math 317    November 5th, 2010

**Name** __________________________________ **Student Number** __________________________________

**No books, notes or calculators are allowed. Unless otherwise stated, to receive full credit you must provide clear, tidy, understandable justification for your solutions.**

| Problem | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 12 | |
| 3 | 8 | |
| 4 | 6 | |
| 5 | 14 | |
| 6 | 10 | |
| Total | 60 | |

**Problem 1 (10 points):**

Let $C$ be the segment of the graph $y = x^2$ from $(0,0)$ to $(\pi, \pi^2)$. Let $f(x,y) = \sqrt{1+4x^2}$. Evaluate $\int_C f ds$. (Note: If your algebra is going wild here, you're doing something wrong.)

**SOLUTION**

A parametrisation of $C$ is:

$$\mathbf{r}(t) = \langle t, t^2 \rangle, \quad 0 \le t \le \pi$$

The derivative is

$$\mathbf{r}'(t) = \langle 1, 2t \rangle, \quad 0 \le t \le \pi$$

Which has magnitude

$$|\mathbf{r}'(t)| = \sqrt{1+4t^2}$$

To calculate the integral:

$$\begin{aligned}
\int_C f ds &= \int_0^\pi f(x(t), y(t)) |r'(t)| dt \\
&= \int_0^\pi \sqrt{1+4t^2}\sqrt{1+4t^2} dt \\
&= \int_0^\pi 1 + 4t^2 dt \\
&= t + \frac{4t^3}{3} |_0^\pi \\
&= \pi + \frac{4\pi^3}{3}
\end{aligned}$$

**Problem 2 (12 points):**
Let $\mathbf{F}(x,y) = \langle \cos(y), 2y - x\sin(y)\rangle$. Let $C$ be a path from $(1,0)$ to $(2,\pi)$. Using the Fundamental Theorem of Line Integrals, evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

**SOLUTION**
The Fundamental Theorem of Line Integrals only applies to conservative fields. We will try to find $f$ such that $\mathbf{F} = \nabla f$. We know

$$P = f_x = \cos(y), \quad Q = f_y = 2y - x\sin(y)$$

Anti-differentiating $P$ with respect to $x$, we find that

$$f = x\cos(y) + g(y)$$

where $g(y)$ is some function of $y$. Taking the partial derivative with respect to $y$, we find

$$f_y = -x\sin(y) + g'(y)$$

Comparing with $Q$, we discover that

$$g'(y) = 2y$$

Anti-differentiating $g'$, we see that

$$g(y) = y^2 + K$$

for some constant $K$. Taking $K = 0$, we find that one possible solution $f$ is

$$f(x,y) = x\cos(y) + y^2$$

(We double-check that this $f$ is correct by verifying that $f_x = P$ and $f_y = Q$.)
From the fundamental theorem of line integrals:

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \nabla f \cdot d\mathbf{r} \\ &= f(2,\pi) - f(1,0) \\ &= 2\cos(\pi) + \pi^2 - (1\cos(0) + 0^2) \\ &= -3 + \pi^2\end{aligned}$$

**Comments on Other Solutions:** Some students computing the line integral directly, against the instructions of the problem to “use the Fundamental Theorem of Line Integrals”. Strictly speaking, that’s disobeying the rules of the problem, but the extra time taken and the extra danger of making an error in calculation (many did!) was enough penalty, and full marks were given for a correct solution of this type. I hope that these students realise, however, that doing it the hard way probably cost them valuable time they could have spent on another problem.

**Problem 3 (8 points, no justification needed):**
Consider the following vector fields:

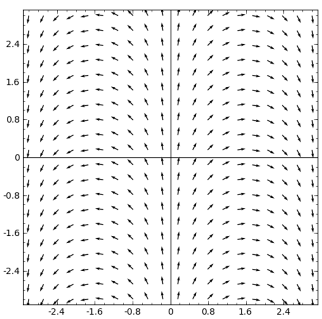
$$\mathbf{F}(x,y) = \langle x, y\rangle, \quad \mathbf{G}(x,y) = \langle -y, x\rangle, \quad \mathbf{H}(x,y) = \langle x, -y\rangle$$

$$\mathbf{I}(x,y) = \langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\rangle, \quad \mathbf{J}(x,y) = \langle \frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}\rangle,$$

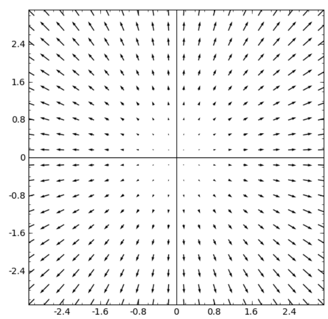
$$\mathbf{K}(x,y) = \langle \sin(x), \cos(x)\rangle, \quad \mathbf{L}(x,y) = \langle \sin(x), \cos(y)\rangle$$

Below each of the following *scaled* vector field graphs, write the name (letter name) of the corresponding vector field. Some letters will not be used.

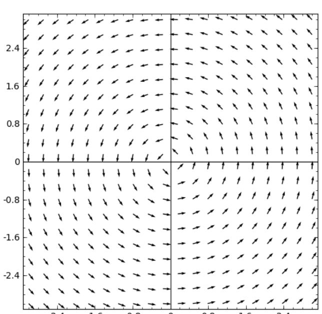
**SOLUTION**



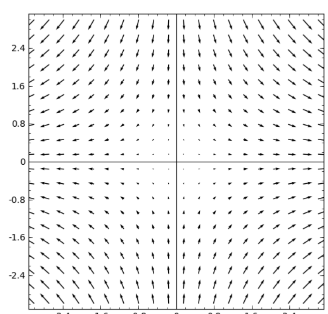
Answer: **K**. This is the only possible answer because this graph shows a field which does not depend on $y$ (each "row" is the same).



Answer: **F**. This is a field you should be familiar with, like an old friend. The vector at a location $(x, y)$ points in direction $\langle x, y\rangle$, i.e. straight out from the origin along the line from the origin to the point $(x, y)$. Points which are farther from the origin have longer vectors (so it cannot be **I**).



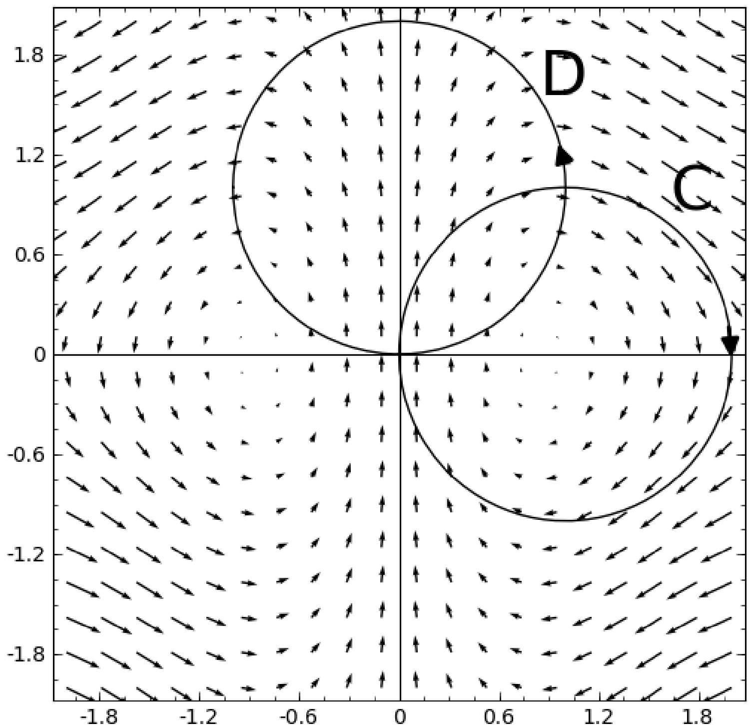
Answer: **J**. This graph shows a field which has vectors of the same length everywhere, which means it is one of **I** or **J**. The vectors are perpendicular to position vector of a point $(x, y)$, so it must be **J**.



Answer: **H**. Comparing to the field above, we see that one could obtain this field from the one above it by flipping each vector in a horizontal line. Alternatively, one can observe that along the $y$ axis vectors point straight toward the origin, while along the $x$ axis they point straight away, and **H** is the only field with that property.

**Problem 4 (6 points, no justification needed):**

Here is a picture of a vector field $\mathbf{F} = \langle P(x,y), Q(x,y)\rangle$, which satisfies $P(-x,y) = -P(x,y)$ and $Q(-x,y) = Q(x,y)$.




The line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ on the unit circle $C$ centred at $(1,0)$ is

(a) $\boxed{\text{positive}}$

(b) negative

(c) zero

(d) not enough information to determine

The vector field "lines up" along the path $C$, i.e. the vectors of $\mathbf{F}$ point to a greater extent in the direction of travel along $C$ than opposite the direction of travel. This means the line integral is positive.

The line integral $\int_D \mathbf{F}\cdot d\mathbf{r}$ on the unit circle $D$ centred at $(0,1)$ is

(a) positive

(b) negative

(c) $\boxed{\text{zero}}$

(d) not enough information to determine

Break $D$ into two paths: $D_1$ travelling from $(0,0)$ to $(0,2)$ along the right semicircle, and $D_2$ travelling from $(0,0)$ to $(2,0)$ along the left semicircle. By the symmetry of the picture (given formally in the preamble), the line integrals along $D_1$ and $D_2$ are equal. So the integral along $D = D_1 - D_2$ is zero.

Is the vector field $\mathbf{F}$ conservative?

(a) Yes

(b) $\boxed{\text{No}}$

By the answer to the first question, this vector field has at least one path $C$ where $\int_C \mathbf{F}\cdot d\mathbf{r} \neq 0$, so it cannot be conservative.

**Problem 5 (3 parts, 14 points):**

Let $\mathbf{F} = \langle 2xy, x^2 + \pi x \rangle$.

a) (2 points) Calculate the curl of the vector field $\mathbf{F}$. (If you do this correctly, your answer should be a constant.)

**SOLUTION**

$$curl = Q_x - P_y = (2x + \pi) - (2x) = \pi$$

b) (2 points) Complete the following statement of Green's Theorem:

Let $C$ be a positively oriented, piecewise-smooth, simple closed curve in the plane and let $D$ be the region bounded by $C$. If $P$ and $Q$ have continuous partial derivatives on an open region that contains $D$, then

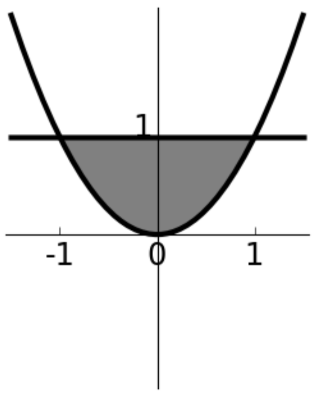
**SOLUTION**

$$\int_C P dx + Q dy = \iint_D Q_x - P_y dA$$

c) (10 points) Let $A$ be the area of the region above $y = x^2$ and below $y = 1$ (shaded in figure below). Using Green's Theorem and part (a), find a constant $K$ and two curves $C_1$ and $C_2$ such that

$$A = K \int_{C_1+C_2} \mathbf{F} \cdot d\mathbf{r}$$

Give the curves by giving a parametrisation (including the domain of $t$). (Note: you will receive 6 points for correct answers, and 4 points for a very clear explanation of why these are correct answers.)




**SOLUTION**

Let $D$ be the shaded region and let $C$ be its boundary, positively oriented.
We apply Green's Theorem to $\mathbf{F}$, and find that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy = \iint_D Q_x - P_y dA = \iint_D \pi dA = \pi \operatorname{Area}(D)$$

Rearranging,

$$\operatorname{Area}(D) = \frac{1}{\pi} \int_C \mathbf{F} \cdot d\mathbf{r}$$

It remains to parametrise the boundary of $D$. It consists of two parts, which we have labelled $C_1$ and $C_2$ above.

**This solution required no integration of any type!!**

$$\boxed{K = \frac{1}{\pi}}$$

$$\boxed{C_1 : \mathbf{r}_1(t) = \langle t, t^2 \rangle, \quad -1 \le t \le 1}$$

$$\boxed{C_2 : \mathbf{r}_2(t) = \langle -t, 1 \rangle, \quad -1 \le t \le 1}$$

**VERY IMPORTANT: There are other correct solutions. Please see the further notes on the next page**

**Important comments:** This problem has many solutions, and many methods of solution, although the one above is by far the simplest. Some students computed the area directly:

$$Area = \int_{-1}^{1}\int_{x^2}^{1} dydx = \int_{-1}^{1} 1 - x^2 dx = x - \frac{x^3}{3}\bigg|_{-1}^{1} = \frac{4}{3}$$

...and then used curves $C_1$ and $C_2$ as above in Green's Theorem to find

$$\int_{C_1+C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1}\int_{x^2}^{1} \pi\, dydx = \pi\int_{-1}^{1} 1 - x^2 dx = \pi\left(x - \frac{x^3}{3}\right)\bigg|_{-1}^{1} = \frac{4\pi}{3}$$

Notice that the work done in both these calculations is the same, with an extra $\pi$ tossed into the second of them! These students therefore discovered that $K = \frac{1}{\pi}$ in a very roundabout way, *if* their calculations didn't go awry in the process. This is in general an inferior solution to the first one, but full marks were given if it was completed correctly, since the extra time taken in the calculations was enough penalty!

Some students had a very creative method, based on the following theme. Compute the area as $\frac{4}{3}$ and then realised that $K$, $C_1$ and $C_2$ could be anything, just so long as

$$K\int_{C_1+C_2} \mathbf{F}\cdot d\mathbf{r} = \frac{4}{3}$$

Why not pick something simple then? Make $C_1$ trivial, make $C_2$ a line segment like $(0,1)$ to $(1,1)$ ($\mathbf{r}(t) = \langle t, 1\rangle, 0 \le t \le 1$), which has

$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{0}^{1} 2tdt = t^2\bigg|_{0}^{1} = 1$$

and let $K = \frac{4}{3}$. Although not as short and sweet as the intended solution, and although it does not use Green's Theorem (as requested by the instructions), this solution demonstrates a solid understanding of *what the question was asking* – this kind of understanding is one of the best skills you can develop in life! It received full marks and these students should be proud of themselves.

**Problem 6 (10 points):**

If a particle moves along a non-trivial (i.e. non-zero length) flow line of a conservative velocity field, can it return to its starting point? If it can, give an example. If it cannot, explain why not. (Note: you will be graded on the clarity of your answer. Writing well counts.)

**SOLUTION A (for those who like equations)**

For a flow line $\mathbf{r}(t)$, the following property holds: $\mathbf{F}(\mathbf{r}(t)) = \mathbf{r}'(t)$. If a particle travelling on a flow line returns to its starting point, then it has travelled on a closed loop flow line, call it $C$. The line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

must be zero by conservativeness of the field. But

$$\mathbf{F} \cdot \mathbf{T} = \mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t))}{|\mathbf{r}'(t)|} = \frac{\mathbf{r}'(t) \cdot \mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t)|^2}{|\mathbf{r}'(t)|} = |\mathbf{r}'(t)| \geq 0$$

Therefore the integral can only come out to 0 if $|\mathbf{r}'(t)| = 0$ always. If so, then $\mathbf{r}'(t) = \mathbf{0}$ for all $t$. This can only happen if curve is a trivial curve, i.e.

$$\mathbf{r}(t) = \mathbf{c}$$

for some constant vector $\mathbf{c}$.

Therefore the only flow lines that are closed loops in a conservative velocity field are trivial ones. And hence a particle travelling on a non-trivial flow line cannot return to its starting point.

**SOLUTION B (for those who prefer words)**

Imagine a particle flowing along a non-trivial flow line. The velocity of the particle is tangent to the flow line in the direction of motion. Furthermore, it is not always zero, since a particle with always-zero velocity doesn't go anywhere, i.e. the flow line would be trivial. The field $\mathbf{F}$ at each point is exactly the velocity of the particle (that's the definition of a flow line). So the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

along any flow line $C$ is positive because the field $\mathbf{F}$ is always in line with the direction of motion of the particle, and is not always zero.

Now suppose that $C$ is a closed loop. In a conservative field, the line integral $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for closed loops $C$.

So if a non-trivial flow line forms a closed loop (i.e. the particle returns to its starting point), then we must conclude that the line integral along that path is both positive (because it is a non-trivial flow line) and zero (because the field is conservative), a contradiction.

**Please see the further comments on the next page.**

**Comments on student solutions.**

Many student had the right idea, but their write-ups were vague, or not well organised. By vague I mean a lack of precision. For example, often students wrote "positive" when they could only conclude "non-negative," and in almost every single case, students did not use the hypothesis that the flow-line was non-trivial: trivial flow-lines *can* return to their starting points, in the sense that they never left them, so a complete proof must *use* the non-trivial condition. Generally missing this point but being fairly well spoken otherwise received a 9/10.

Among those students who did not seem to grasp the main point, the principal misconceptions seemed to be:

(a) Many didn't remember what a flow line was. The definition is a curve $\mathbf{r}(t)$ such that $\mathbf{F}(\mathbf{r}(t)) = \mathbf{r}'(t)$.

(b) Students confused velocity fields and force fields. A vector field is considered a velocity field when its vectors are taken to represent velocities. A vector field is considered a force field when its vectors are taken to represent forces. In this problem, we are considering the field a velocity field, and a flow line is any path of a particle which has *velocity* dictated by the field (interpreted as velocity) at each location.

(c) The earth's orbit is not a flow line. This relates to the last item. The earth's orbit is a path in which the particle (the planet) has *acceleration* dictated by the field interpreted as force at each location. Compare to the last item.

(d) Use of the term "work". Strictly speaking, "work" is a term from physics. It refers to the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ *in the case that the field is interpreted as a force field.* I fear I was not sufficiently clear about this in lecture.

(e) Some students believed that

$$\mathbf{r}' = \nabla f \implies \mathbf{r} = f$$

by some sort of anti-differentiation. While I think (especially in context) that this shows creativity of the type needed to solve problems of this sort (and that's a good thing!) it also shows imprecise thinking. The quantity $\mathbf{r}$ is a vector, while $f$ is a scalar, so the second equation doesn't make sense. Also, the "prime" (') in $\mathbf{r}'$ is a quite different kind of derivative than the gradient $\nabla$, so you can't 'undo' them both with the same process.