the origin. [This is a special case of Gauss's Law (Equation 16.7.11) for a single charge. The relationship between ε and ε_0 is $\varepsilon = 1/(4\pi\varepsilon_0)$.]

Another application of the Divergence Theorem occurs in fluid flow. Let $\mathbf{v}(x, y, z)$ be the velocity field of a fluid with constant density ρ. Then $\mathbf{F} = \rho\mathbf{v}$ is the rate of flow per unit area. If $P_0(x_0, y_0, z_0)$ is a point in the fluid and B_a is a ball with center P_0 and very small radius a, then $\operatorname{div}\mathbf{F}(P) \approx \operatorname{div}\mathbf{F}(P_0)$ for all points in B_a since $\operatorname{div}\mathbf{F}$ is continuous. We approximate the flux over the boundary sphere S_a as follows:

$$\iint_{S_a} \mathbf{F}\cdot d\mathbf{S} = \iiint_{B_a} \operatorname{div}\mathbf{F}\,dV = \iiint_{B_a} \operatorname{div}\mathbf{F}(P_0)\,dV = \operatorname{div}\mathbf{F}(P_0)V(B_a)$$

This approximation becomes better as $a \to 0$ and suggests that

$$\boxed{8} \qquad \operatorname{div}\mathbf{F}(P_0) = \lim_{a\to 0} \frac{1}{V(B_a)} \iint_{S_a} \mathbf{F}\cdot d\mathbf{S}$$

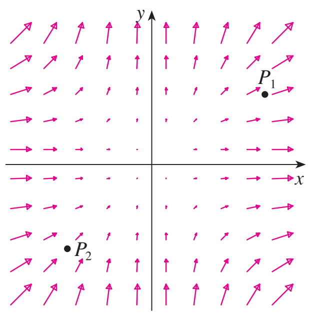


FIGURE 4
The vector field $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j}$

Equation 8 says that $\operatorname{div}\mathbf{F}(P_0)$ is the net rate of outward flux per unit volume at P_0. (This is the reason for the name *divergence*.) If $\operatorname{div}\mathbf{F}(P) > 0$, the net flow is outward near P and P is called a **source**. If $\operatorname{div}\mathbf{F}(P) < 0$, the net flow is inward near P and P is called a **sink**.

For the vector field in Figure 4, it appears that the vectors that end near P_1 are shorter than the vectors that start near P_1. Thus the net flow is outward near P_1, so $\operatorname{div}\mathbf{F}(P_1) > 0$ and P_1 is a source. Near P_2, on the other hand, the incoming arrows are longer than the outgoing arrows. Here the net flow is inward, so $\operatorname{div}\mathbf{F}(P_2) < 0$ and P_2 is a sink. We can use the formula for $\mathbf{F}$ to confirm this impression. Since $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j}$, we have $\operatorname{div}\mathbf{F} = 2x + 2y$, which is positive when $y > -x$. So the points above the line $y = -x$ are sources and those below are sinks.

16.9 EXERCISES

1–4 Verify that the Divergence Theorem is true for the vector field $\mathbf{F}$ on the region E.

1. $\mathbf{F}(x, y, z) = 3x\,\mathbf{i} + xy\,\mathbf{j} + 2xz\,\mathbf{k}$,
   E is the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$
2. $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + xy\,\mathbf{j} + z\,\mathbf{k}$,
   E is the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane
3. $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$,
   E is the solid cylinder $x^2 + y^2 \leqslant 1$, $0 \leqslant z \leqslant 1$
4. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$,
   E is the unit ball $x^2 + y^2 + z^2 \leqslant 1$

5–15 Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$; that is, calculate the flux of $\mathbf{F}$ across S.

5. $\mathbf{F}(x, y, z) = e^x \sin y\,\mathbf{i} + e^x \cos y\,\mathbf{j} + yz^2\,\mathbf{k}$,
   S is the surface of the box bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 2$
6. $\mathbf{F}(x, y, z) = x^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + xz^4\,\mathbf{k}$,
   S is the surface of the box with vertices $(\pm1, \pm2, \pm3)$
7. $\mathbf{F}(x, y, z) = 3xy^2\,\mathbf{i} + xe^z\,\mathbf{j} + z^3\,\mathbf{k}$,
   S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$
8. $\mathbf{F}(x, y, z) = x^3y\,\mathbf{i} - x^2y^2\,\mathbf{j} - x^2yz\,\mathbf{k}$,
   S is the surface of the solid bounded by the hyperboloid $x^2 + y^2 - z^2 = 1$ and the planes $z = -2$ and $z = 2$
9. $\mathbf{F}(x, y, z) = xy \sin z\,\mathbf{i} + \cos(xz)\,\mathbf{j} + y \cos z\,\mathbf{k}$,
   S is the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$
10. $\mathbf{F}(x, y, z) = x^2y\,\mathbf{i} + xy^2\,\mathbf{j} + 2xyz\,\mathbf{k}$,
    S is the surface of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + 2y + z = 2$
11. $\mathbf{F}(x, y, z) = (\cos z + xy^2)\,\mathbf{i} + xe^{-z}\,\mathbf{j} + (\sin y + x^2z)\,\mathbf{k}$,
    S is the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$
12. $\mathbf{F}(x, y, z) = x^4\,\mathbf{i} - x^3z^2\,\mathbf{j} + 4xy^2z\,\mathbf{k}$,
    S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = x + 2$ and $z = 0$
13. $\mathbf{F}(x, y, z) = 4x^3z\,\mathbf{i} + 4y^3z\,\mathbf{j} + 3z^4\,\mathbf{k}$,
    S is the sphere with radius R and center the origin

14. $\mathbf{F} = \mathbf{r}/|\mathbf{r}|$, where $r = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$,
S consists of the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and the disk $x^2 + y^2 \leqslant 1$ in the xy-plane

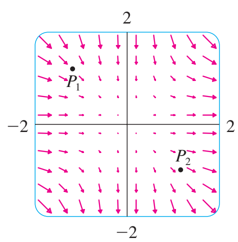
CAS **15.** $\mathbf{F}(x, y, z) = e^y \tan z\,\mathbf{i} + y\sqrt{3 - x^2}\,\mathbf{j} + x \sin y\,\mathbf{k}$,
S is the surface of the solid that lies above the xy-plane and below the surface $z = 2 - x^4 - y^4$, $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$

CAS **16.** Use a computer algebra system to plot the vector field $\mathbf{F}(x, y, z) = \sin x \cos^2 y\,\mathbf{i} + \sin^3 y \cos^4 z\,\mathbf{j} + \sin^5 z \cos^6 x\,\mathbf{k}$ in the cube cut from the first octant by the planes $x = \pi/2$, $y = \pi/2$, and $z = \pi/2$. Then compute the flux across the surface of the cube.

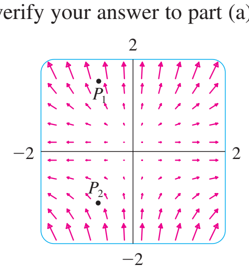
17. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where
$\mathbf{F}(x, y, z) = z^2 x\,\mathbf{i} + \left(\frac{1}{3}y^3 + \tan z\right)\mathbf{j} + (x^2 z + y^2)\,\mathbf{k}$
and S is the top half of the sphere $x^2 + y^2 + z^2 = 1$.
[*Hint:* Note that S is not a closed surface. First compute integrals over S_1 and S_2, where S_1 is the disk $x^2 + y^2 \leqslant 1$, oriented downward, and $S_2 = S \cup S_1$.]

18. Let $\mathbf{F}(x, y, z) = z \tan^{-1}(y^2)\,\mathbf{i} + z^3 \ln(x^2 + 1)\,\mathbf{j} + z\,\mathbf{k}$.
Find the flux of $\mathbf{F}$ across the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$ and is oriented upward.

19. A vector field $\mathbf{F}$ is shown. Use the interpretation of divergence derived in this section to determine whether div $\mathbf{F}$ is positive or negative at P_1 and at P_2.



20. (a) Are the points P_1 and P_2 sources or sinks for the vector field $\mathbf{F}$ shown in the figure? Give an explanation based solely on the picture.
(b) Given that $\mathbf{F}(x, y) = \langle x, y^2 \rangle$, use the definition of divergence to verify your answer to part (a).



CAS **21–22** Plot the vector field and guess where div $\mathbf{F} > 0$ and where div $\mathbf{F} < 0$. Then calculate div $\mathbf{F}$ to check your guess.

21. $\mathbf{F}(x, y) = \langle xy, x + y^2 \rangle$

22. $F(x, y) = \langle x^2, y^2 \rangle$

23. Verify that div $\mathbf{E} = 0$ for the electric field $\mathbf{E}(\mathbf{x}) = \dfrac{\varepsilon Q}{|\mathbf{x}|^3}\,\mathbf{x}$.

24. Use the Divergence Theorem to evaluate $\iint_S (2x + 2y + z^2)\,dS$ where S is the sphere $x^2 + y^2 + z^2 = 1$.

25–30 Prove each identity, assuming that S and E satisfy the conditions of the Divergence Theorem and the scalar functions and components of the vector fields have continuous second-order partial derivatives.

25. $\displaystyle\iint_S \mathbf{a} \cdot \mathbf{n}\,dS = 0$, where $\mathbf{a}$ is a constant vector

26. $\displaystyle V(E) = \frac{1}{3}\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

27. $\displaystyle\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$

28. $\displaystyle\iint_S D_{\mathbf{n}} f\,dS = \iiint_E \nabla^2 f\,dV$

29. $\displaystyle\iint_S (f\nabla g) \cdot \mathbf{n}\,dS = \iiint_E (f\nabla^2 g + \nabla f \cdot \nabla g)\,dV$

30. $\displaystyle\iint_S (f\nabla g - g\nabla f) \cdot \mathbf{n}\,dS = \iiint_E (f\nabla^2 g - g\nabla^2 f)\,dV$

31. Suppose S and E satisfy the conditions of the Divergence Theorem and f is a scalar function with continuous partial derivatives. Prove that

$$\iint_S f\mathbf{n}\,dS = \iiint_E \nabla f\,dV$$

These surface and triple integrals of vector functions are vectors defined by integrating each component function. [*Hint:* Start by applying the Divergence Theorem to $\mathbf{F} = f\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant vector.]

32. A solid occupies a region E with surface S and is immersed in a liquid with constant density ρ. We set up a coordinate system so that the xy-plane coincides with the surface of the liquid and positive values of z are measured downward into the liquid. Then the pressure at depth z is $p = \rho g z$, where g is the acceleration due to gravity (see Section 6.5). The total buoyant force on the solid due to the pressure distribution is given by the surface integral

$$\mathbf{F} = -\iint_S p\mathbf{n}\,dS$$

where $\mathbf{n}$ is the outer unit normal. Use the result of Exercise 31 to show that $\mathbf{F} = -W\mathbf{k}$, where W is the weight of the liquid displaced by the solid. (Note that $\mathbf{F}$ is directed upward because z is directed downward.) The result is *Archimedes' principle:* The buoyant force on an object equals the weight of the displaced liquid.