16.5 EXERCISES

1–8 Find (a) the curl and (b) the divergence of the vector field.

1. $\mathbf{F}(x, y, z) = xyz\,\mathbf{i} - x^2y\,\mathbf{k}$

2. $\mathbf{F}(x, y, z) = x^2yz\,\mathbf{i} + xy^2z\,\mathbf{j} + xyz^2\,\mathbf{k}$

3. $\mathbf{F}(x, y, z) = \mathbf{i} + (x + yz)\,\mathbf{j} + \left(xy - \sqrt{z}\right)\mathbf{k}$

4. $\mathbf{F}(x, y, z) = \cos xz\,\mathbf{j} - \sin xy\,\mathbf{k}$

5. $\mathbf{F}(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$

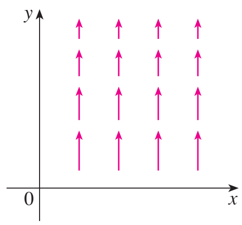
6. $\mathbf{F}(x, y, z) = e^{xy}\sin z\,\mathbf{j} + y\tan^{-1}(x/z)\,\mathbf{k}$

7. $\mathbf{F}(x, y, z) = \langle \ln x, \ln(xy), \ln(xyz) \rangle$

8. $\mathbf{F}(x, y, z) = \langle e^x, e^{xy}, e^{xyz} \rangle$

9–11 The vector field **F** is shown in the xy-plane and looks the same in all other horizontal planes. (In other words, **F** is independent of z and its z-component is 0.)
(a) Is div **F** positive, negative, or zero? Explain.
(b) Determine whether curl $\mathbf{F} = \mathbf{0}$. If not, in which direction does curl **F** point?

9.



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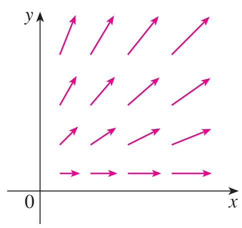
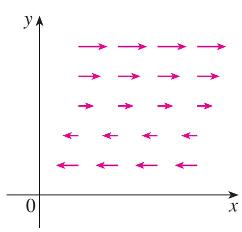


11.



12. Let f be a scalar field and **F** a vector field. State whether each expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field.
(a) curl f
(b) grad f
(c) div **F**
(d) curl(grad f)
(e) grad **F**
(f) grad(div **F**)
(g) div(grad f)
(h) grad(div f)
(i) curl(curl **F**)
(j) div(div **F**)
(k) (grad f) × (div **F**)
(l) div(curl(grad f))

13–18 Determine whether or not the vector field is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

13. $\mathbf{F}(x, y, z) = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$

14. $\mathbf{F}(x, y, z) = xyz^2\,\mathbf{i} + x^2yz^2\,\mathbf{j} + x^2y^2z\,\mathbf{k}$

15. $\mathbf{F}(x, y, z) = 2xy\,\mathbf{i} + (x^2 + 2yz)\,\mathbf{j} + y^2\,\mathbf{k}$

16. $\mathbf{F}(x, y, z) = e^z\,\mathbf{i} + \mathbf{j} + xe^z\,\mathbf{k}$

17. $\mathbf{F}(x, y, z) = ye^{-x}\,\mathbf{i} + e^{-x}\,\mathbf{j} + 2z\,\mathbf{k}$

18. $\mathbf{F}(x, y, z) = y\cos xy\,\mathbf{i} + x\cos xy\,\mathbf{j} - \sin z\,\mathbf{k}$

19. Is there a vector field **G** on $\mathbb{R}^3$ such that curl $\mathbf{G} = \langle x\sin y, \cos y, z - xy \rangle$? Explain.

20. Is there a vector field **G** on $\mathbb{R}^3$ such that curl $\mathbf{G} = \langle xyz, -y^2z, yz^2 \rangle$? Explain.

21. Show that any vector field of the form

$$\mathbf{F}(x, y, z) = f(x)\,\mathbf{i} + g(y)\,\mathbf{j} + h(z)\,\mathbf{k}$$

where f, g, h are differentiable functions, is irrotational.

22. Show that any vector field of the form

$$\mathbf{F}(x, y, z) = f(y, z)\,\mathbf{i} + g(x, z)\,\mathbf{j} + h(x, y)\,\mathbf{k}$$

is incompressible.

23–29 Prove the identity, assuming that the appropriate partial derivatives exist and are continuous. If f is a scalar field and **F**, **G** are vector fields, then $f\mathbf{F}$, $\mathbf{F}\cdot\mathbf{G}$, and $\mathbf{F}\times\mathbf{G}$ are defined by

$$(f\mathbf{F})(x, y, z) = f(x, y, z)\,\mathbf{F}(x, y, z)$$
$$(\mathbf{F}\cdot\mathbf{G})(x, y, z) = \mathbf{F}(x, y, z)\cdot\mathbf{G}(x, y, z)$$
$$(\mathbf{F}\times\mathbf{G})(x, y, z) = \mathbf{F}(x, y, z)\times\mathbf{G}(x, y, z)$$

23. $\operatorname{div}(\mathbf{F} + \mathbf{G}) = \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}$

24. $\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl}\mathbf{F} + \operatorname{curl}\mathbf{G}$

25. $\operatorname{div}(f\mathbf{F}) = f\operatorname{div}\mathbf{F} + \mathbf{F}\cdot\nabla f$

26. $\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}\mathbf{F} + (\nabla f)\times\mathbf{F}$

27. $\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot\operatorname{curl}\mathbf{F} - \mathbf{F}\cdot\operatorname{curl}\mathbf{G}$

28. $\operatorname{div}(\nabla f \times \nabla g) = 0$

29. $\operatorname{curl}(\operatorname{curl}\mathbf{F}) = \operatorname{grad}(\operatorname{div}\mathbf{F}) - \nabla^2\mathbf{F}$

30–32 Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $r = |\mathbf{r}|$.

30. Verify each identity.
(a) $\nabla\cdot\mathbf{r} = 3$
(b) $\nabla\cdot(r\mathbf{r}) = 4r$
(c) $\nabla^2r^3 = 12r$

31. Verify each identity.
(a) $\nabla r = \mathbf{r}/r$ (b) $\nabla \times \mathbf{r} = \mathbf{0}$
(c) $\nabla(1/r) = -\mathbf{r}/r^3$ (d) $\nabla \ln r = \mathbf{r}/r^2$

32. If $\mathbf{F} = \mathbf{r}/r^p$, find div $\mathbf{F}$. Is there a value of p for which div $\mathbf{F} = 0$?

33. Use Green's Theorem in the form of Equation 13 to prove **Green's first identity**:

$$\iint_D f\nabla^2 g\, dA = \oint_C f(\nabla g)\cdot \mathbf{n}\, ds - \iint_D \nabla f \cdot \nabla g\, dA$$

where D and C satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous. (The quantity $\nabla g \cdot \mathbf{n} = D_{\mathbf{n}}\, g$ occurs in the line integral. This is the directional derivative in the direction of the normal vector $\mathbf{n}$ and is called the **normal derivative** of g.)

34. Use Green's first identity (Exercise 33) to prove **Green's second identity**:

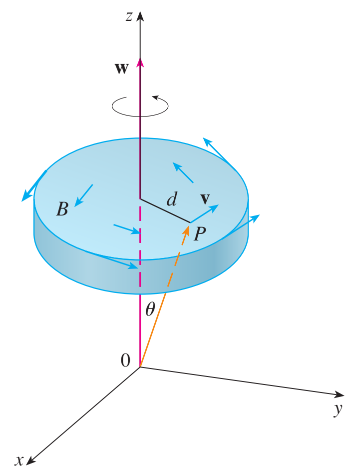
$$\iint_D (f\nabla^2 g - g\nabla^2 f)\, dA = \oint_C (f\nabla g - g\nabla f)\cdot \mathbf{n}\, ds$$

where D and C satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous.

35. Recall from Section 14.3 that a function g is called *harmonic* on D if it satisfies Laplace's equation, that is, $\nabla^2 g = 0$ on D. Use Green's first identity (with the same hypotheses as in Exercise 33) to show that if g is harmonic on D, then $\oint_C D_{\mathbf{n}}\, g\, ds = 0$. Here $D_{\mathbf{n}}\, g$ is the normal derivative of g defined in Exercise 33.

36. Use Green's first identity to show that if f is harmonic on D, and if $f(x, y) = 0$ on the boundary curve C, then $\iint_D |\nabla f|^2\, dA = 0$. (Assume the same hypotheses as in Exercise 33.)

37. This exercise demonstrates a connection between the curl vector and rotations. Let B be a rigid body rotating about the z-axis. The rotation can be described by the vector $\mathbf{w} = \omega\mathbf{k}$, where ω is the angular speed of B, that is, the tangential speed of any point P in B divided by the distance d from the axis of rotation. Let $\mathbf{r} = \langle x, y, z\rangle$ be the position vector of P.
(a) By considering the angle θ in the figure, show that the velocity field of B is given by $\mathbf{v} = \mathbf{w} \times \mathbf{r}$.
(b) Show that $\mathbf{v} = -\omega y\,\mathbf{i} + \omega x\,\mathbf{j}$.
(c) Show that curl $\mathbf{v} = 2\mathbf{w}$.



38. Maxwell's equations relating the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ as they vary with time in a region containing no charge and no current can be stated as follows:

$$\text{div}\,\mathbf{E} = 0 \qquad \text{div}\,\mathbf{H} = 0$$

$$\text{curl}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \qquad \text{curl}\,\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

where c is the speed of light. Use these equations to prove the following:

(a) $\nabla \times (\nabla \times \mathbf{E}) = -\dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{E}}{\partial t^2}$

(b) $\nabla \times (\nabla \times \mathbf{H}) = -\dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{H}}{\partial t^2}$

(c) $\nabla^2 \mathbf{E} = \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{E}}{\partial t^2}$ [*Hint:* Use Exercise 29.]

(d) $\nabla^2 \mathbf{H} = \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{H}}{\partial t^2}$

39. We have seen that all vector fields of the form $\mathbf{F} = \nabla g$ satisfy the equation curl $\mathbf{F} = \mathbf{0}$ and that all vector fields of the form $\mathbf{F} = \text{curl}\,\mathbf{G}$ satisfy the equation div $\mathbf{F} = 0$ (assuming continuity of the appropriate partial derivatives). This suggests the question: Are there any equations that all functions of the form $f = \text{div}\,\mathbf{G}$ must satisfy? Show that the answer to this question is "No" by proving that *every* continuous function f on $\mathbb{R}^3$ is the divergence of some vector field. [*Hint:* Let $\mathbf{G}(x, y, z) = \langle g(x, y, z), 0, 0\rangle$, where $g(x, y, z) = \int_0^x f(t, y, z)\, dt$.]