Math 205A Final Exam (75 points)

Name: Solutions.

- Check that you have 8 questions on four pages.
- Show all your work to receive full credit for a problem.

1. (12 points) Short answers: (Show all the calculations to get the answers. No explanations needed.)

   (a) If C is a 4×5 matrix, what is the largest possible rank of C? What is the smallest possible dimension of Nul C?

   Max. number of pivots $= 4$. So largest rank $= 4$.
   Number of columns $=$ rank C + dim Nul C.
   $5 = 4 + \dim \text{Nul } C$.
   So smallest dim Nul $C = 1$.

   (b) For a 3×3 matrix B, $\det B = -1$. Find $\det 4B$.

   $\det 4B = 4^3 \det B$ (Since each row is scaled by 4 and there are three rows.)
   $= 64(-1)$
   $\det 4B = -64$

   (c) Find the distance between the vector $\vec{u} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

   Distance $= \| \bar{u} - \bar{v} \|$
   $= \left\| \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right\| = \sqrt{16+0} = 4$.

   (d) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

   $$T(x_1, x_2, x_3) = (x_1 - x_2, 2x_2 - x_3).$$

   Find a matrix A such that $T(\vec{x}) = A\vec{x}$.

   $$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ 2x_2 - x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
   $$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

   So $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$.

(e) Let $\vec{p_1}(t) = 1$, $\vec{p_2}(t) = 2t$, $\vec{p_3}(t) = 4t^2 - 2$ and $\vec{p_4}(t) = 8t^3 - 12t$. Then $\mathcal{B} = \{\vec{p_1}, \vec{p_2}, \vec{p_3}, \vec{p_4}\}$ is a basis for $\mathbb{P}_3$. Find the polynomial $\vec{q}$ in $\mathbb{P}_3$, given that $[q]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}$.

$$\bar{q} = 2\bar{p}_1 - 3\bar{p}_2 + 1\bar{p}_3 + 0 \cdot \bar{p}_4$$

$$= 2(1) - 3(2t) + (4t^2 - 2) + 0$$

$$= 2 - 6t + 4t^2 - 2$$

$$\bar{q} = -6t + 4t^2$$

2. (4 points) Suppose the columns of a 4×4 matrix A span $\mathbb{R}^4$. Is $\det A = 0$? Explain.

Since the columns of A span $\mathbb{R}^4$,
there is a pivot in every row of A.
Since A is a square matrix, there is a pivot in every column as well.
Thus $A \sim I$ i.e A is invertible.

So $\det A \neq 0$.

3. (12 points) Determine if the following sets are subspaces of the appropriate vector spaces. If a set is a subspace, find a basis and the dimension of the subspace. If a set is not a subspace, explain why.

(a) $W = \left\{ \text{all vectors in } \mathbb{R}^3 \text{ of the form } \begin{bmatrix} r+5s \\ s-r-3t \\ 2r+5t \end{bmatrix} \text{ where r, s, t are in } \mathbb{R} \right\}$.

$$\begin{bmatrix} r+5s \\ s-r-3t \\ 2r+5t \end{bmatrix} = r\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + s\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix}$$

So $W = \text{Span}\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}}_{\bar{v}_1}, \underbrace{\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}_2}, \underbrace{\begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix}}_{\bar{v}_3} \right\}$. Hence W is a subspace of $\mathbb{R}^3$.

$$\begin{bmatrix} 1 & 5 & 0 \\ -1 & 1 & -3 \\ 2 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix}$$

From this, we see that the eqn. $c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = \bar{0}$ has a nontrivial soln and $\bar{v}_3$ is a linear combination of $\bar{v}_1$ and $\bar{v}_2$. So $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is not a lin. ind. set but $\{\bar{v}_1, \bar{v}_2\}$ is a lin. ind. set that spans W.

Hence basis for $W = \{\bar{v}_1, \bar{v}_2\}$. dim $W = 2$.

(b) $W = \{\text{all polynomials in } \mathbb{P}_2 \text{ of the form } \vec{p}(t) = at + bt^2, \text{ where a, b are integers}\}$.

$\bar{p}(t) = 2t + t^2$ is in W.

$c = \frac{1}{3}$ is a scalar.

$c \cdot \bar{p}(t) = \frac{2}{3}t + \frac{1}{3}t^2$ is not in W because

$\frac{2}{3}$ and $\frac{1}{3}$ are not integers.

So W is not closed under scalar multiplication.

So W is not a subspace of $\mathbb{P}_2$.

(c) $W = \{$all symmetric matrices in $M_{2\times 2}\}$. (Recall that a symmetric matrix is a matrix A such that $A^T = A$.)

A symmetric matrix in $M_{2\times2}$ has the form $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

where a, b, c are ~~r~~real numbers.

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{A} + b\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{B} + c\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{C}.$$

A, B, C are in W and every matrix in W can be written as a linear combination of A, B, C.

So $W = \text{Span}\{A, B, C\}$. Hence W is a subspace of $M_{2\times2}$.

$M_{2\times2}$ is ~~isomorp~~ isomorphic to $\mathbb{R}^4$ and under the standard coordinate mapping, A goes to $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, B goes to $\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$, C goes to $\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$.

$\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$ is a lin. ind. set in $\mathbb{R}^4$. So $\{A, B, C\}$ is a lin. ind. set in $M_{2\times2}$. So basis for $W = \{A, B, C\}$. $\dim W = 3$.

4. (5 points) Suppose A is a symmetric $n \times n$ matrix and B is any $n \times n$ matrix. Explain why BAB^T is orthogonally diagonalizable.

$$(BAB^T)^T = (B^T)^T \cdot A^T B^T \text{ (properties of transpose)}$$
$$= BA^TB^T \text{ (properties of transpose)}$$
$$= BAB^T \text{ (} A^T = A \text{ since } A \text{ is a symmetric matrix)}$$

$(BAB^T)^T = BAB^T$. So BAB^T is a symmetric matrix.

Hence it is orthogonally diagonalizable.

5. (12 points) Define a linear transformation $T : \mathbb{P}_1 \to \mathbb{R}_2$ by $T(\vec{p}) = \begin{bmatrix} \vec{p}(1) \\ \vec{p}(1) \end{bmatrix}$. (Recall that a vector $\vec{p}$ in $\mathbb{P}_1$ is a polynomial of the form $a + bt$.)

(a) Find $T(3)$ and $T(2 - 7t)$.

$\bar{p}(t) = 3$. $\bar{p}(1) = 3$. So $T(\bar{p}) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

$\bar{q}(t) = 2 - 7t$. $\bar{q}(1) = 2 - 7 = -5$. So $T(\bar{q}) = \begin{bmatrix} -5 \\ -5 \end{bmatrix}$

(b) Find a polynomial p in $\mathbb{P}_1$ such that $T(\vec{p}) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ or explain why we cannot find such a polynomial.

Let $\bar{p} = a + bt$ $T(\bar{p}) = \begin{bmatrix} \bar{p}(1) \\ \bar{p}(1) \end{bmatrix} = \begin{bmatrix} a+b \\ a+b \end{bmatrix}$. We want this to equal $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$. So $a + b = 4$. Thus, many answers are possible. One answer is $a = 1, b = 3$. $\bar{p}(t) = 1 + 3t$.

Another answer is $a = \frac{1}{2}, b = \frac{7}{2}$. $\bar{p}(t) = \frac{1}{2} + \frac{7}{2}t$.

(c) Find a polynomial that spans the kernel of T. (Recall that the kernel of T is the space of all vectors that are mapped to the zero vector under T,i.e., the kernel is the null space of T.)

$\bar{p} = a + bt$. $T(\bar{p}) = \bar{0}$ gives $\begin{bmatrix} \bar{p}(1) \\ \bar{p}(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ie $\begin{bmatrix} a+b \\ a+b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

So $a + b = 0$ or $b = -a$.

Thus, $\bar{p} = a - at = a(1 - t)$. Hence kernel of $T = \text{Span}\{1 - t\}$.

(d) Is T one-to-one? Explain.

We saw in part (c) that $T(\bar{p}) = \bar{0}$ has infinitely many solutions. Thus T is not one-to-one.

6. (12 points) Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

(a) Is 1 an eigenvalue of A? If so, find a basis for the eigenspace corresponding to 1. If not, explain why not.

$(A-I)\bar{x}=\bar{0}$ $A-I=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $x_1=-x_2-x_3$ x_2, x_3 free. $\bar{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -x_2-x_3 \\ x_2 \\ x_3 \end{bmatrix}$

$=x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}+x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

The eqn. $(A-I)\bar{x}=\bar{0}$ has infinitely many solutions. So 1 is an eigenvalue of A.

Basis for eigenspace $=\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ (The set is lin. ind. and spans the eigenspace.)

(b) Is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ an eigenvector of A? If so, find the corresponding eigenvalue. If not, explain why not.

$A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}=4\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

So $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A and the corresponding eigenvalue is 4.

(c) Use your answers in parts (a) and (b) to diagonalize A, if possible. (That is, if possible, find matrices P and D such that $A = PDP^{-1}$.) If not, explain why not.

$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ are eigenvectors of A.

$\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ We have a pivot in every column. So these eigenvectors are lin. ind.

Since A has three lin. ind. eigenvectors, A is diagonalizable.

$P=\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $D=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

7. (10 points) Hurricanes develop low pressure at their centers that generates high winds. The maximum wind speed s (in knots) and the central pressure p of a hurricane are approximately related by the equation $b_0 + b_1 p = s$. We have the following data on four recent Atlantic hurricanes in the United States.

p	905	920	960	990
s	130	110	80	60

Find b_0 and b_1 so that the model $b_0 + b_1 p = s$ is a least-squares fit to the data. (Start by using the given data to write a system of linear equations to determine b_0 and b_1.)

$$b_0 + b_1(905) = 130 \qquad A = \begin{bmatrix} 1 & 905 \\ 1 & 920 \\ 1 & 960 \\ 1 & 990 \end{bmatrix}, \quad \bar{s} = \begin{bmatrix} 130 \\ 110 \\ 80 \\ 60 \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}.$$

$$b_0 + b_1(920) = 110$$

$$b_0 + b_1(960) = 80$$

$$b_0 + b_1(990) = 60.$$

To find b_0, b_1 we solve the eqn. $A\bar{x} = \bar{s}$.

$$[A \ \bar{s}] \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \to 0 = 1.$$ So the system is inconsistent.

So we use the normal eqns. $A^TA\bar{x} = A^T\bar{s}$ to find the least-squares soln.

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 905 & 920 & 960 & 990 \end{bmatrix} \begin{bmatrix} 1 & 905 \\ 1 & 920 \\ 1 & 960 \\ 1 & 990 \end{bmatrix} = \begin{bmatrix} 4 & 3775 \\ 3775 & 3567125 \end{bmatrix}.$$

$$A^T\bar{s} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 905 & 920 & 960 & 990 \end{bmatrix} \begin{bmatrix} 130 \\ 110 \\ 80 \\ 60 \end{bmatrix} = \begin{bmatrix} 380 \\ 355050 \end{bmatrix}$$

Now we solve the eqn. $A^TA\bar{x} = A^T\bar{s}$.

$$\begin{bmatrix} 4 & 3775 & 380 \\ 3775 & 3567125 & 355050 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 850 \\ 0 & 1 & -0.8 \end{bmatrix}$$

So $b_0 = 850$, $b_1 = -0.8$

8. (8 points) Let $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. Let $W = \text{Span}\{\vec{u}_1, \vec{u}_2\}$ and $\vec{y} = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$.

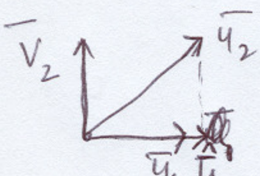
(a) The set $\{\vec{u}_1, \vec{u}_2\}$ is not an orthogonal set. Find two vectors in W that are orthogonal to each other and span W. (Use the vectors $\vec{u}_1$ and $\vec{u}_2$ to produce the two orthogonal vectors.)

Let $\bar{v}_1 = \bar{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$$\hat{u}_2 = \text{proj}_{\bar{u}_1} \bar{u}_2 = \left(\frac{\bar{u}_2 \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1}\right) \bar{u}_1 = \frac{2}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

Let $\bar{v}_2 = \bar{u}_2 - \hat{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$$\bar{v}_1 = \bar{u}_1, \quad \bar{v}_2 = \bar{u}_2 - \hat{u}_2 = \bar{u}_2 - \left(\frac{\bar{u}_2 \cdot \bar{u}_1}{\bar{u}_1 \cdot \bar{u}_1}\right) \bar{u}_1.$$



Thus, $\bar{v}_1, \bar{v}_2$ are linear combinations of $\bar{u}_1, \bar{u}_2$.

So $W = \text{Span}\{\bar{u}_1, \bar{u}_2\} = \text{Span}\{\bar{v}_1, \bar{v}_2\}$.

Also, $\bar{v}_1 \cdot \bar{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$. So they are orthogonal to each other.

(b) Find a vector in W that is closest to $\vec{y}$.

Vector in W that is closest to $\bar{y}$

$= \hat{y}$

$= \left(\frac{\bar{y} \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1}\right) \bar{v}_1 + \left(\frac{\bar{y} \cdot \bar{v}_2}{\bar{v}_2 \cdot \bar{v}_2}\right) \bar{v}_2$ We use $\bar{v}_1, \bar{v}_2$ because they are orthogonal to each other.

$= \frac{-1}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{3}{1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$. So $\hat{y} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$.