4. Let $\mathbb{F}$ be a field of size κ and let V be an $\mathbb{F}$-vector space of infinite dimension λ.

(a) Show that the dimension of the dual space V^* is κ^λ.

(b) Use the Main Theorem of Cardinal Arithmetic to simplify $\dim_{\mathbb{F}}(V^*)$ in the case where $|\mathbb{F}| = \beth_{\omega_1}$ and $\dim_{\mathbb{F}}(V) = \aleph_0$.

Proof. (a) Let $B = \{b_i : i < \lambda\}$ be an ordered basis for V. By the hypothesis, $|B| = \lambda$ and $|\mathbb{F}| = \kappa$. Each $f \in V^* = \text{Hom}_{\mathbb{F}}(V, \mathbb{F})$ is uniquely determined by its restriction $f|_B : B \to \mathbb{F}$, which is arbitrary. Hence, $|V^*| = |\mathbb{F}^B| = \kappa^\lambda$.

Claim: If the $\dim_{\mathbb{F}}(V)$ is infinite, then $\dim_{\mathbb{F}}(V^*)$ is equal to the cardinality of V^*.

It is clear $\dim_{\mathbb{F}}(V^*) \leq |V^*|$.

Case 1: ($|\mathbb{F}| < |V^*|$) Let W be an infinite $\mathbb{F}$ vector space. We argue if $|\mathbb{F}| < |W|$, then $\dim(W) = |W|$. Let C be a basis for W. Note if $\mathbb{F}$ and C are finite, then W would be finite. Hence $|\mathbb{F}| + |C|$ must be infinite. Since every vector in W is expressible as a finite string of symbols from $\mathbb{F} \cup C \cup \{+\}$, we must have $|W| \leq |\mathbb{F}| + |C| + \aleph_0 = |\mathbb{F}| + |C|$. Since $|\mathbb{F}| < |W| \leq |\mathbb{F}| + |C|$ and the right hand side is infinite, $|\mathbb{F}| < |C|$.

Putting these ideas together,

$$|C| \leq |W| \leq |\mathbb{F}| + |C| = \max(|\mathbb{F}|, |C|) = |C|.$$

Hence $|W| = |C| = \dim(W)$.

Case 2: ($|\mathbb{F}| = |V^*|$) Choose $a \in \mathbb{F}$ and define a functional $\hat{a} \in V^*$ by,

$$\hat{a}(b_i) = \left\{ \begin{array}{ll} a^i, & \text{if } i < \omega \\ 0, & \text{else} \end{array} \right\} \tag{1}$$

for $b_i \in B$.

Let's argue that $\hat{\mathbb{F}} = \{\hat{a} : a \in \mathbb{F}\}$ is an independent subset of V^*. If not, there exists a dependence relation, $c_1\hat{a}_1 + \cdots + c_k\hat{a}_k = 0$ where not all $c_j = 0$ for $1 \leq j \leq k$. Writing this dependence relation in matrix form,

$$A\vec{c} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_k \\ a_1^2 & a_2^2 & \cdots & a_k^2 \\ \vdots & & & \vdots \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix}.$$

Because the columns of a Vandermonde matrix are independent, the vector $\vec{c} = \vec{0}$. This contradicts our assumption of not all $c_j = 0$.

The $\dim_{\mathbb{F}}(V^*)$ is greater than or equal to the size of any independent subset, like $\hat{\mathbb{F}}$. Then, $|\mathbb{F}| = |\hat{\mathbb{F}}| \leq \dim_{\mathbb{F}}(V^*) \leq |V^*| = |\mathbb{F}|$. Hence, $\dim_{\mathbb{F}}(V^*) = |V^*|$.

Therefore, since $|V^*| = \kappa^\lambda$ and $\dim_{\mathbb{F}}(V^*) = |V^*|$, $\dim_{\mathbb{F}}(V^*) = \kappa^\lambda$.

(b) Allow $\kappa = \beth_{\omega_1}$ and $\lambda = \aleph_0$. Since $\lambda < \kappa$, so we are not in *Case 1* of the Main Theorem of Cardinal Arithmetic. Since κ is not λ-reachable from below, we are not in *Case 2*. Also note the cofinality of $\beth_{\omega_1}$ is ω_1 and $\omega_1 > \aleph_0$. By the Main Theorem of Cardinal Arithmetic,

$$\begin{aligned}\dim_{\mathbb{F}}(V^*) &= \beth_{\omega_1}^{\aleph_0} \\ &= \beth_{\omega_1}.\end{aligned}$$

□