3. Show that GCH is equivalent over ZFC to the statement that for infinite cardinals κ, it holds that $\kappa^+ = \kappa^{cf(\kappa)}$:

Proof. We have the following chain of inequalities, the nontrivial one resulting from Konig's theorem:

$$cf(\kappa) \leq \kappa < \kappa^+ \leq \kappa^{cf(\kappa)} \leq \kappa^\kappa = 2^\kappa$$

From the Generalized Continuum Hypothesis, we have $\kappa^+ = 2^\kappa$, and thus $\kappa = \kappa^{cf(\kappa)}$. This takes care of the forward direction of the proof.

For the backward direction of the proof, assume that $\kappa^+ = \kappa^{cf(\kappa)}$.

Case 1: κ is regular. If κ is regular, then we would have $\kappa^{cf(\kappa)} = \kappa^\kappa = 2^\kappa$, so GCH follows easily.

Case 2: κ is singular. If κ is singular, then we need to show that $\kappa^{cf(\kappa)} = \kappa^\kappa$

Assume, for contradiction that λ is the least infinite cardinal such that $\lambda^+ < 2^\lambda$. We will argue that $\lambda^+ < \lambda^{\mathrm{cf}(\lambda)}$, contradicting the assumption in the first sentence of this paragraph.

Choose an increasing sequence of cardinals

$$\mu_0 < \mu_1 < \mu_2 < \ldots$$

of length $\mathrm{cf}(\lambda)$ whose union is λ.

Because $\mu_i < \lambda$ for all i, this means that $2^{\mu_i} = (\mu_i)^+ \leq \mu_{i+1} < \lambda$, so

$$\bigcup \mu_i = \bigcup 2^{\mu_i} = \lambda$$

or, in arithmetic terms,

$$\sum \mu_i = \sum 2^{\mu_i} = \lambda$$

We then have the following chain of inequalities, where the first inequality is our assumption for contradiction:

$$\lambda^+ < 2^\lambda = 2^{\sup(\mu_i)} \leq 2^{\sum \mu_i} = \prod 2^{\mu_i} \leq \prod \lambda = \lambda^{\mathrm{cf}(\lambda)}$$

In partiuclar, we have $\lambda^+ < \lambda^{\mathrm{cf}(\lambda)}$, a contradiction of the hypothesis of this problem. □