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Set Theory
Assignment 1

1. Show that the following statement is equivalent over ZFC to CH:

The real plane $\mathbb{R}^2$ may be partitioned into two cells, $\{A, B\}$, in such a way that any horizontal line in $\mathbb{R}^2$ has countable intersection with A and any vertical line in $\mathbb{R}^2$ has countable intersection with B.

Proof. ($\Rightarrow$) Assume CH so that $\aleph_1 = 2^{\aleph_0}$. Let $(r_\alpha)_{\alpha<\aleph_1}$ be an enumeration of $\mathbb{R}$. Consider the partition of $\mathbb{R}^2$ whose cells are $A = \{(r_\alpha, r_\beta) \in \mathbb{R}^2 | \alpha \leq \beta < \aleph_1\}$ and the complement B. Let us denote L_{y_0} as the horizontal line $y = y_0$ in $\mathbb{R}^2$ and L_{x_0} as the vertical line $x = x_0$ in $\mathbb{R}^2$. Given $y_0 = r_\beta$, we have

$$L_{y_0} \cap A = \{(r_0, y_0), (r_1, y_0), \dots, (r_\beta, y_0)\},$$

so $|L_{y_0} \cap A| \leq |\beta| < \aleph_1$. Similarly, for any x_0, $|L_{x_0} \cap B| \leq |\beta| < \aleph_1$. In both cases, by CH, both intersections are countable.

($\Leftarrow$) Assume CH does not hold. Then $|\mathbb{R}| > \aleph_1$, i.e. $|\mathbb{R}| \geq \aleph_2$. Let's start by looking at a set H of $\aleph_1$ many vertical lines. Suppose, for contradiction, we have a partition $\{A, B\}$ satisfying the above conditions. We have $|(\bigcup H) \cap B| = |\bigcup(\{L \cap B \mid L \in H\})| = \aleph_0 \cdot \aleph_1 = \aleph_1$, because there are exactly $\aleph_1$ many vertical lines, and B has countable intersection with each of them. Since $|(\bigcup H) \cap B| < |\mathbb{R}|$, the projection of $(\bigcup H) \cap B$ onto the vertical axis is not surjective. Therefore there must exist some $y_0 \in \mathbb{R}$ such that the entire line $y = y_0$ contains a set K, the set of intersection points of the line $y = y_0$ and the vertical lines of H, with $\aleph_1$-many points in H and not in B. Thus, $K \subseteq A$, which contradicts the ability for A to have countable intersection with every horizontal line.

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