Problem 3. Show the following:

(a) Suppose that κ is an infinite cardinal and α is a smaller ordinal. Show that the interval $[\alpha, \kappa)$ of ordinals is order-isomorphic to κ.

Proof. Suppose that $[\alpha, \kappa)$ is enumerated by an ordinal γ. There are two things we need to prove:

(1) Using γ to enumerate $[\alpha, \kappa)$ in an order-preserving way gives us that $\gamma \leq \kappa$;

(2) In fact, $\gamma = \kappa$.

For (1), we create a bijection $\gamma \to [\alpha, \kappa)$ and by transfinite induction we have that $f(x) \geq x$. We can explicitly define this bijection f by

$$f(0) = \alpha$$

$$f(S(x)) = S(f(x))$$

$$f(\lambda) = \bigcup_{\alpha < \lambda} f(\alpha)$$

For (2), we can assume for contradiction that $\gamma < \kappa$. Then $\gamma + \alpha = \max(\gamma, \alpha) < \kappa$, which is a contradiction.

□

(b) Suppose that $\kappa_0 < \kappa_1 < \kappa_2 < \cdots$ is a strictly increasing sequence of cardinals with limit κ. Explain why each interval $[\kappa_i, \kappa_{i+1})$ is order-isomorphic to κ_{i+1}.

Proof. Follows from part (a): plug in κ_i for $|\alpha|$ and κ_{i+1} for κ in the statement of (a). □

(c) Suppose that $\kappa_0 < \kappa_1 < \kappa_2 < \cdots$ is an increasing sequence of cardinals with limit κ. Show that $\sum \kappa_i = \sup\{\kappa_i\} = \kappa$.

Proof. Assume the sequence $\kappa_0 < \kappa_1 < \kappa_2 < \cdots$ is increasing with limit κ. Then by part (b), we can write

$$[0, \kappa) = [0, \kappa_0) \sqcup [\kappa_0, \kappa_1) \sqcup [\kappa_1, \kappa_2) \sqcup \cdots$$
$$\Rightarrow \kappa = \kappa_0 + \kappa_1 + \kappa_2 + \cdots$$
$$\Rightarrow \sup\{\kappa_i\} = \sum \kappa_i.$$

□