Problem 2. (a) Show that $|\mathcal{P}(\aleph_0)| = 2^{\aleph_0}$.

(b) Show that the ordered set $\langle \mathcal{P}(\aleph_0); \subseteq \rangle$ contains a chain of cardinality $2^{\aleph_0}$.

(c) Show that the ordered set $\langle \mathcal{P}(\aleph_0); \subseteq \rangle$ contains an antichain of cardinality $2^{\aleph_0}$.

Proof. (a) Define $f : \mathcal{P}(\aleph_0) \to 2^{\aleph_0}$ by $f(A) = \chi_A$ for $A \subseteq \aleph_0$, where

$$\chi_A(a) = \begin{cases} 1 & \text{if } a \in A, \\ 0 & \text{otherwise} \end{cases}.$$

Let $A, B \subseteq \aleph_0$. By the Axiom of Extensionality, $\chi_A = \chi_B$ implies $A = B$. Moreover, by the Axiom of Separation, $f(G) = g$, where $G = \{x \in \aleph_0 \mid g(x) = 1\}$ is a set. Thus, f is a bijection, proving $|\mathcal{P}(\aleph_0)| = |2^{\aleph_0}| = 2^{\aleph_0}$.

(b) Let $h : \mathbb{N} \to \mathbb{Q}$ be a bijection, which exists since $|\mathbb{N}| = |\mathbb{Q}|$. Define

$$D(r) = \{n \in \aleph_0 \mid h(n) < r\}, \tag{1}$$

where r is a real number. Observe that $r \leq r'$ implies $D(r) \subseteq D(r')$ for any reals r, r' since $h(n) < r$ implies $h(n) < r'$. Thus, in general, $D(r) \subseteq D(r')$ or $D(r') \subseteq D(r)$. Moreover, $D(r) = D(r')$ implies $h(n) < r \iff h(n) < r'$ for any $n \in \aleph_0$, hence by the Archimedean Property $r = r'$. Therefore, the set

$$D = \{L \in \mathcal{P}(\aleph_0) \mid L = D(r)\} \tag{2}$$

is a chain of cardinality $2^{\aleph_0}$.

(c) Let h and $D(r)$ be defined as above, and let

$$L(r) = \{n \in \aleph_0 \mid r < h(n)\}, \tag{3}$$
$$E(r) = D(r) \cup L(r+1) \tag{4}$$

for any real number r. Suppose r, r' are real numbers, and without loss of generality, assume $r < r'$. Then, by the Archimedean property, there is some $q \in \mathbb{Q}$ satisfying $r < q < r'$. Thus, $D(r') \backslash D(r)$ and $L(r+1) \backslash L(r'+1)$ are non-empty, so $E(r) \not\subseteq E(r')$ and $E(r') \not\subseteq E(r)$.

Moreover, if $E(r) = E(r')$, then because h is a bijection, $D(r) = D(r')$ and $L(r) = L(r')$, hence $r = r'$. Thus,

$$F = \{H \in \mathcal{P}(\aleph_0) \mid H = E(r)\} \tag{5}$$

is an antichain of cardinality $2^{\aleph_0}$.

□