Problem 1. Find all pairs (κ, λ) of cardinals such that the cardinal sum $\kappa +_c \lambda$ agrees with the ordinal sum $\kappa +_o \lambda$.

Lemma 1. For all cardinals $\kappa < \lambda$, where λ is infinite, $\kappa +_o \lambda = \lambda$.

Proof. Let $(W_1, <_1), (W_2, <_2)$ be well-ordered sets with order type κ and λ, respectively. By Hrbacek and Jech, Theorem 6.5.3, the sum $(W, <)$ of $(W_1, <_1)$ and $(W_2, <_2)$ is order isomorphic to $\kappa +_o \lambda$. Since $\kappa < \lambda$, $(W_1, <_1), (W_2, <_2)$ have different order types, and if W_2 is isomorphic to an initial segment of W_1, $\lambda \leq \kappa$, contradicting $\kappa < \lambda$. The fact that W_2 is not isomorphic to an initial segment of W_1 implies that W_1 is isomorphic to an initial segment of W_2, according to Theorem 4.1.3. in Hrbacek and Jech. Thus, W has order type λ, hence $\kappa +_o \lambda = \lambda$. □

Lemma 2. For all infinite cardinals $\kappa \geq \lambda$, $\kappa +_c \lambda \neq \kappa +_o \lambda$.

Proof. Suppose $\kappa +_c \lambda = \kappa +_o \lambda$. By the Axiom of Choice, $\kappa = \kappa +_c \lambda$. By Theorem 6.5.3. in Hrbacek and Jech, κ is an initial segment of $\kappa +_o \lambda$. Because $\lambda > 0$, this must be a proper initial segment. Moreover, because there is an initial segment of $\kappa +_o \lambda$ that has the same cardinality as this sum, $\kappa +_o \lambda$ is not an initial ordinal. Thus, $\kappa +_o \lambda$ is not a cardinal, so $\kappa +_c \lambda = \kappa \neq \kappa +_o \lambda$. □

Theorem 3. For all cardinals κ, λ, $\kappa +_c \lambda = \kappa +_o \lambda$ if and only if both κ, λ are finite or $\kappa < \lambda$, with λ infinite.

Proof. We proved in class that the cardinal sum of finite cardinals agrees with the ordinal sum. When one of κ, λ are infinite, this theorem is proven by Lemmas 1-3 above. □