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2. Let T be the theory axiomatized by all the axioms of ZFC except the Axiom of Foundation. (T = ZFC $\setminus$ {Fnd}.). From T, prove that the Axiom of Foundation is equivalent to the following statement:

There is no function f with domain ω such that $f(n+1) \in f(n)$ for all n.

Axiom of Foundation: If A is a nonempty set, then there exists a set $B \in A$ such that $A \cap B = \emptyset$.

Proof.

($\Rightarrow$) Assume there is a function with domain ω such that $f(n+1) \in f(n)$. By the Axiom of Replacement, $S = \{z | (\exists k)(k \in \omega \wedge f(k) = z)\}$ is a set. But then for any $n \in \omega$, $f(n+1) \in f(n) \cap S \neq \emptyset$, contradicting the Axiom of Foundation.

($\Leftarrow$) Assume that the Axiom of Foundation does not hold. So for some set A, $B \cap A \neq \emptyset$ for all $B \in A$. Fix an element B of A. By the Axiom of Choice, there is a choice function $c : \mathcal{P}(A) \setminus \{\emptyset\} \to A$. By the Recursion Theorem, there exists a unique function $g : \omega \to \mathcal{P}(A)$ such that $g(0) = c(B)$ and $g(n+1) = c(g(n) \cap A)$. But then $g(n+1) \in g(n)$ for all $n \in \omega$, contradicting the original assumption.

□