Oscar Bender-Stone
Robert Green

1. We have explained why the Russell class $\mathcal{R} = \{x | x \notin x\}$ is a proper class. Show that each one of the following is also a proper class.

1. the class $\mathcal{C}$ of all sets.
2. the class $\mathcal{D}$ of all 1-element sets.

   (You should express both $\mathcal{C}$ and $\mathcal{D}$ as classes, and then that they are proper.)

1. $\mathcal{C} = \{x | x = x\}$

   Proof. Assume $\mathcal{C}$ is a set. Then $\mathcal{C} \in \mathcal{C}$. But this contradicts the Axiom of Foundation because $\mathcal{C} \cap \{\mathcal{C}\}$ is nonempty. □

   Alternative proof:

   Proof. Then by the Axiom of Separation, we can predicate on $\mathcal{C}$ and form the subset $\{x \in \mathcal{C} | x \notin x\} = \mathcal{R} \subseteq \mathcal{C}$. But we have already established that $\mathcal{R}$ is not a set. □

2. $\mathcal{D} = \{x | \exists y \forall z (z \in x \iff z = y)\}$

   Proof. Assume $\mathcal{D}$ is a set. Then $\{\mathcal{D}\} \in \mathcal{D}$. By the Axiom of Pairing, $y = \{\mathcal{D}, \{\mathcal{D}\}\}$ is a set. But then $\{\mathcal{D}\} \in \mathcal{D} \cap y$ and $\mathcal{D} \in \{\mathcal{D}\} \cap y$, which contradicts the Axiom of Foundation.

   □